FINITE ELEMENT MODELING OF STREAMFLOW ROUTING

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ABSTRACT

Application of the finite element methods to the unsteady flow equations is presented in predicting the depth, velocity and volumetric flow rate in a stream. Two simplified models--the explicit and weighted-implicit finite element models--are proposed as methods for streamflow routing. The equation of continuity that conserves the mass of flow is completely retained, while the equation of momentum that conserves the flow momentum is modified to account only for the friction and gravity terms.

The objective of this study is to develop an efficient, simple model that is capable of accurately predicting discharges with least or no stability problems in computations. The weighted-implicit kinematic model has been developed and tested for a flood routing problem. The results agree closely with previously found solution.

INTRODUCTION

The process of determining the water depths, velocities, and discharges in the channels, streams or reservoirs under unsteady conditions arising from flood motions is commonly referred to as flood routing. Interest in flood routing and, in part, in unsteady flows in water resources stems from the need to plan, design, regulate, and manage our flood-prone areas and other water resources systems. Following the development of the complete one-dimensional partial differential equations of unsteady flow in open channels, known as the "Saint Venant equations" named after Barre' de Saint[4], a French mathematician who first derived them in 1871, there has been an on-going need to develop the most efficient and accurate methods of solving these equations.

Early attempts to provide solutions of the complete unsteady flow equations were impractical until the advent of computers. The digital computer development has elicited extensive study in the use of numerical analysis for obtaining accurate and computationally feasible solutions to the continuity and momentum equations of the unsteady flow for which no analytical solution exist. Various schemes of the finite difference methods, such as the explicit, characteristics and implicit, have taken their turns in the computations. However, numerical properties in the form of convergence, accuracy, and stability have limited the use of each of the finite-difference schemes.

A survey of previous literature indicates that many investigators to date have employed the finite-difference schemes in flood routing problems. Isaccson et al.[17] investigated flood routing in their pioneering work in the Ohio River. Amein and Fang[3] used an implicit scheme in solving the streamflow routing problem in natural channels in North Carolina. Amein[1,2] also used the method of characteristics to solve the streamflow problem in an attempt to studying the effects of friction on peak flows. Pinder and Sauer[20] employed the explicit method in simulating the flood wave modification due to bank storage effects. Fread[9,10,11,12,13,14] investigated the routing problems using the implicit four-point and the weighted four-point finite-difference schemes. Chaudhry and Contractor[5], Henderson[16], Liggett amd Woolheser[19] also used finite difference methods to solve for approximate flow equations.

Another prospect to the solutions of the unsteady flow equations in using the finite element methods. The application of the method to the water resources system in general is more recent. As Chung[7] indicated, the application of the method to fluid flow was initiated by Zienkiewicz[25] in 1965, following the pioneering work of Turner, Clough, Martin, and Topp in 1956 who presented the first paper on the subject. Few documented reports on the use of the finite element methods to flood routing are based on methods of weight residuals[8,18,21] in that the Galerkin's principle predominates. This is because the differential equations defining the differential equations defining the unsteady flow in open channels are not self-adjoint. The variational principles can not be applied. In fact, the finite element method, mostly the weighted residual approach coupled with the use of digital computer has rapidly become one of the most powerful tools in solving the complex engineering problems of continuous media.

In this study, the Galerkin weighted residuals method was utilized in transforming the governing partial differential equations of unsteady open channel flow into the ordinary differential equivalents. The explicit and weighted-implicit kinematic models were subsequently developed from the system of ordinary differential equations. Both models were applied seperately to a rectangular channel similar to the type investigated by Viessman et al.[24] for the purpose of exploring the basic principles as well as the mathematical aspects of the methods. The simulated results compared favorable with those of Viessman et al., and the observed difference resides on the speed and stability. In this regard, the weighted-implicit kinematic model excels.

Use of the simple models in terms of computer storage and cost will continue to be favored provided good engineering judgement is exercised on where to apply them. Due to this reason, these models will receive enormous attention in spite of the added emphasis given to the complete solutions of the unsteady flow equations.

MATHEMATICAL MODEL

The distribution of discharge, depth of flow, and velocity of flow in a stream are represented in Figure 1. The mathematical model that predicts the discharge on a space and time basis can be represented by the following equations.

Conservation of mass:

$$\frac{\partial y}{\partial t} + y\,\frac{\partial v}{\partial x} + v\,\frac{\partial y}{\partial x} - q(x,t) = 0 \qquad (1)$$

Conservation of momentum:

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} + g\,\frac{\partial y}{\partial x} + \frac{v}{y}\,q(x,t) - g(s_o - s_f) = 0 \qquad (2)$$

where

v - the average velocity of flow, ft. per sec.

y - the depth of flow, ft.

q - the lateral inflow in the channel reach, Δx, ft. per sec.

x - the distance along channel reach, ft.

t - the time, sec.

g - the gravitational acceleration , ft. per sq. sec.

s_o - the slopeof the river bottom

s_f - the friction slope ($= \frac{n^2\ v^2}{2.22R^{4/3}}$)

n - Manning's roughness coefficient

R - the hydraulic radius, ft.

The two dependent variables in Equations (1) and (2) are the depth of flow, y(x,t), and the velocity of flow, v(x,t). The channel

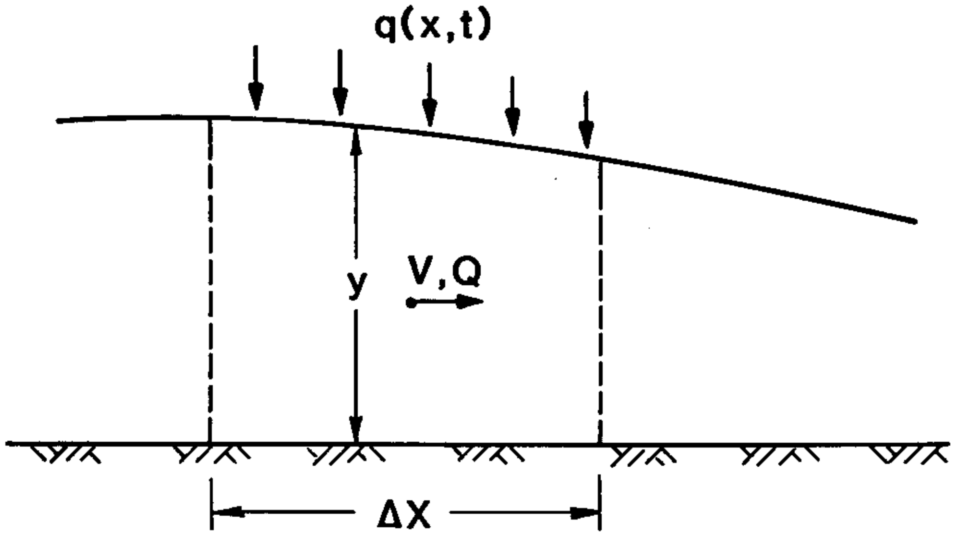

Figure 1. Streamflow Element.

geometry is specified by the area of flow, $A(x)$, the hydraulic width, $B(x)$, (where $A(x) = B(x)\ \partial y/\partial x$), and the slope $s_o = s_o(x)$. The lateral inflow $q(x,t)$ has about three possible sources of contribution, namely, the rainfall on the stream $RF(x,t)$, overland flow $q_o(x,t)$, and the subsurface inflow $q_s(x,t)$. The usual unit of $q(x,t)$ is cubic feet per day per feet of channel length; whereas $RF(x,t)$ term is given in feet per day.

The equations of the conservation of mass and momentum presented above are classified as one-dimensional in the sense that flow characteristics such as the depth and velocity are considered to vary only in the longitudinal x-direction of the channel. Other simplifying assumptions inherent in their derivation are as follows[6,12,15]: (1) the velocity is constant and the water surface is horizontal across any section perpendicular to the longitudinal axis; (2) the flow is gradually varied with hydrostatic pressure prevailing at all points in the flow such that the vertical acceleration of water particles may be neglected; (3) the longitudinal axis of the channel can be approximated by a straight line; (4) the bottom slope of the channel is small; (5) the bed of the channel is fixed, i.e., no scouring or deposition is assumed to occur; (6) the resistance coefficient for steady uniform turbulent flow is considered applicable, and an empirical resistance equation such as the Manning equation describes the resistance effects; and (7) the flow is incompressible and homogeneous in density.

Once the velocity of flow and depth of flow are computed from Equations (1) and (2), the discharge can be computed from the following equation:

$$Q = v\ y \tag{3}$$

where Q - the stream flow volumetric flow rate; cubic feet per sec per channel width of flow.

KINEMATIC FLOW MODELS

The simplified version of the momentum equation, which neglects pressure and inertia terms as compared with friction and gravity terms coupled with continuity equation herein called the 'Kinematic Flow Model', is proposed and solved in two routines. The explicit and weighted-implicit solution techniques were applied, each predicting the depth of flow, velocity of flow, and discharge in the stream.

The explicit kinematic flow model in its final form is linear and consequently the solution is sought by a direct method similar to the tridiagonal matrix alborithm set-up by Varga.[23] Solution proceeds by matrix reduction similar to Gaussian elimination. In contrast to the explicit scheme, the weighted-implicit scheme yields a set of non-linear ordinary differential equations which are solved by the Newton-Raphson iterative method.

VERIFICATION OF MODELS

The computer coded routines of the explicit and weighted-implicit kinematic flow models were applied individually to simulate the hypothetical flood in a prismatic rectangular channel presented by Viessman et al[24] using the explicit finite-difference scheme. The problem considers a 20-ft. wide rectangular channel, 2 mi. long, having uniform flow of 6-ft. depth, subject to an upstream increase in flow of 2000 cfs in a period of 20 minutes. This flow then decreases uniformly to the initial flow depth in an additional period of 40 minutes. The channel has a bottom slope of 0.0015 ft/ft and an estimated Manning's coefficient of 0.02.

Similar to Viessman et al, the distance step of 528 feet was used throughout the simulation although the two routines have the option to accept a variable distance step. Also the weighted-implicit routine has a built-in option to route flood in a trapezoidal, triangular, or rectangular channel geometry. For the first two geometries, the right- and left-side slopes captioned ZRS and ZLS should have assigned values other than zeros. The triangular geometry will have zero width for input value.

The spatial and temporal distributions of flow predicted by the explicit and weighted-implicit kinematic models are plotted in Figure 2 along with those of Viessman et al. The kinematic flow models depicted slight attenuations in the peak flows at midstream as well as the downstream locations. This performance is acceptable since the longitudinal

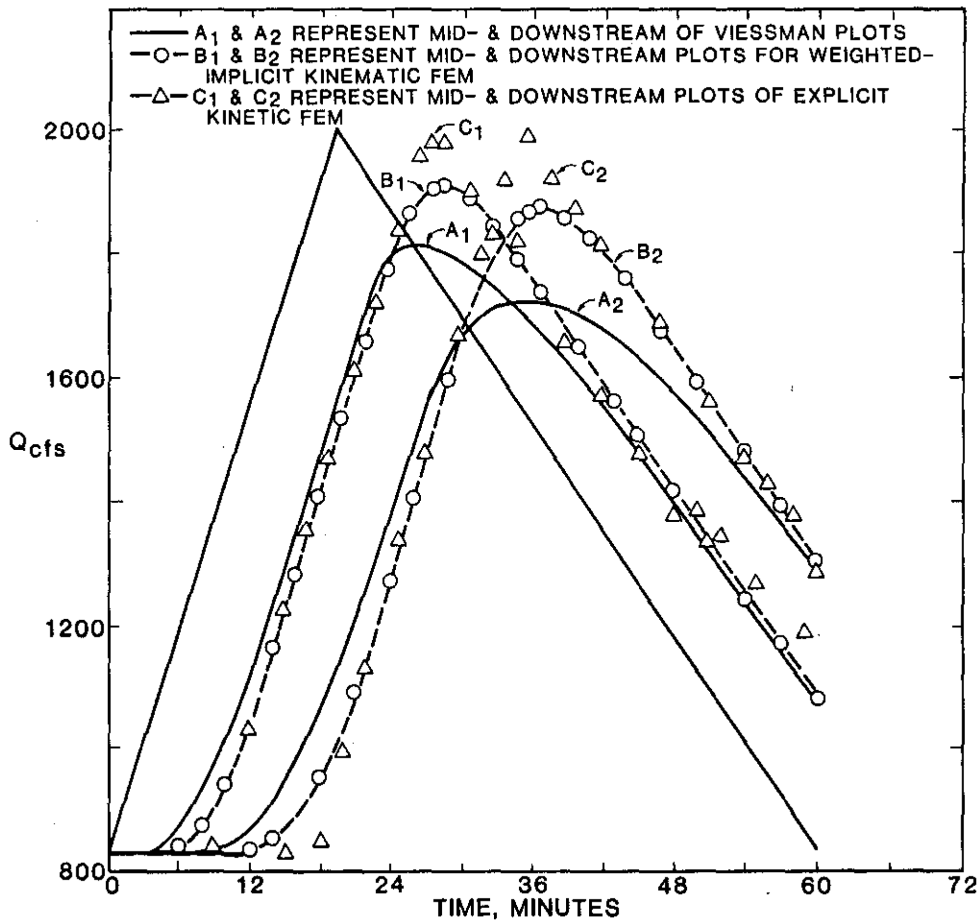

Figure 2. Comparison of Discharge Hydrographs for Explicit and Weighted-Implicit FEM and Viessman Solution.

channel slope utilized for the simulation falls within the bulk-pack of 10 feet per mile (10.3%) for which the use of kinematic approximation is justified. Other situations for which the kinematic model could be applied with great success is the overland flow as well as to the slow-rising flood waves.

While the explicit finite element model is limited to a time step of 2 seconds because of stability problems, the weighted-implicit model is unconditionally stable. The weighted-implicit finite model was run for Δt = 180, 300 and 600 seconds with various values of weighting factor θ, such as 0.55, 0.75 and 1.0, respectively. Fastest convergence was obtained with $\theta = 1.0$ and it is recommended for use with this routine. It is no surprise that $\theta = 1.0$ affords rapid convergence because the scheme becomes fully implicit. Figure 3 compares the predicted flows for $\theta = 0.75$ and $\theta = 1.0$ with time step of 300 seconds. The flow results are in agreement.

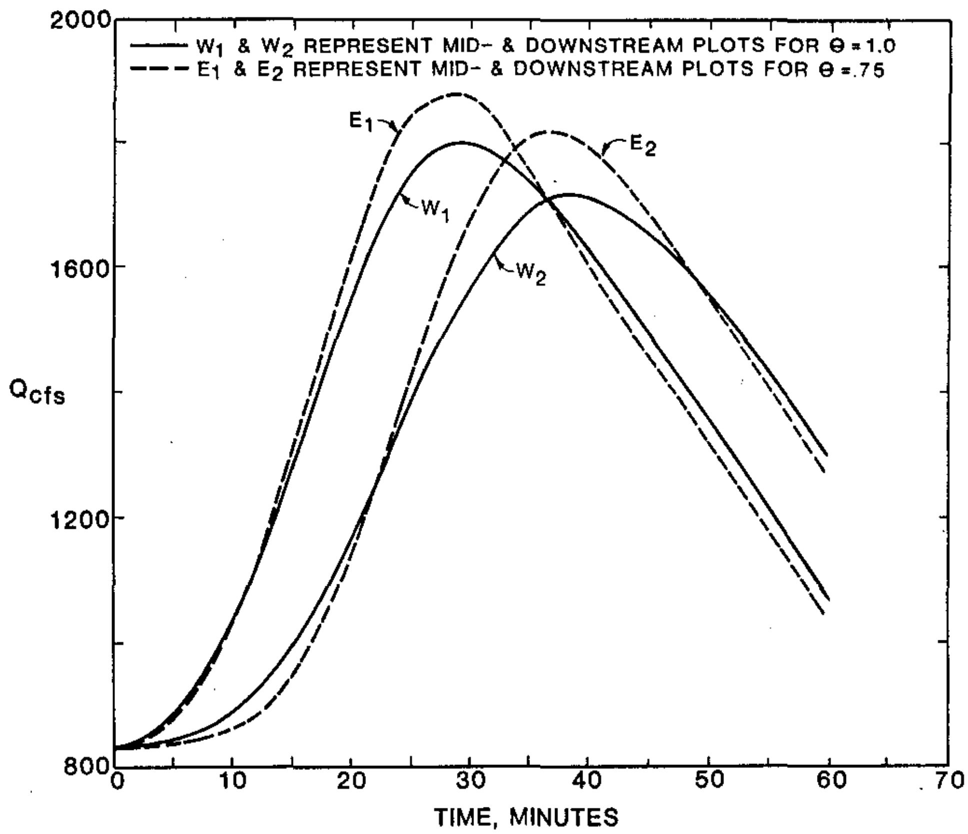

Figure 3. Weighted-Implicit FEM Simulation at 300 Seconds for Θ of 0.75 and 1.00.

DISCUSSION OF RESULTS

The numerical properties of the kinematic finite element flow models, such as the rate of convergence, accuracy and stability, need to be assessed through well established mathematical relations. For instance, the courant condition is employed in the explicity finite-difference technique to evaluate the dynamic stability condition arising from the size of the time steps. Since similar conditions in the finite element technique are not versatile and few in use are formulated under limited assumptions, we are therefore encouraged to draw comparisons from documentations established for the finite difference schemes, at least for the time being.

The convergence criteria is herein taken as a condition in which the solution of the finite element equation for a finite grid size approaches the true solution of the original partial differential equation. For the weighted-implicit finite difference scheme proposed by Fread[11], the convergence criteria was developed by determining the functional form of the truncation error through the Taylor's series expansion about the point at which the difference equation is computed. The truncations error can be expressed as

$$\text{Truncation Error, TR} = (2\theta-1)\ O(\Delta t)+O(\Delta t^2)+O(\Delta x^2)$$

where O indicates "order of" when $\theta = 1$, the truncation error, $TR = O(\Delta t)+O(\Delta t^2)+O(\Delta x^2)$, which shows that the fully implicit difference scheme is only first order accurate due to the t term. However, when $\theta = 0.5$, the error shows a second order accuracy for Δt and Δx.

The weighted-implicit kinematic finite element model converges to the true solution for various values of the weighting factor ranging from 0.55 to 1.0. For θ less than 0.55, the model is completely unstable and invariably does not converge. This leads to the concept of numerical stability, defined as a condition whereby the numerical round-off errors introduced in the computational procedure fails to be amplified into an unlimited error. If errors generated at time level $(t+\Delta t)$ is smaller than the errors at time t and not vice versa, solution is said to be stable.

The stability of the non-linear difference equations of the Saint Venant is sometimes investigated by fourier analysis[10,11]. This analysis is known as the Von Neumann method. In general, results indicate that an implicit difference formulation of the unsteady flow equation is unconditionally stable for any ratio of $\Delta x/\Delta t$, when the θ weighting factor is restricted to the range $0.5 < \theta < 1.0$. The analysis prove that stability of the implicit difference equation does not depend on the ratio $\Delta x/\Delta t$ as do the explicit and characteristic methods.

The weighted-implicit kinematic finite element routine is found to be unconditionally stable for the weighting factor in the range of $0.55 \leq \theta \leq 1.0$. Rapid convergence coupled with stability requirement makes a weighting factor of unity best choice for this model.

CONCLUSIONS

1. Explicit and weighted-implicit kinematic flow models have been developed to predict the velocity of flow, depth of flow, and discharge in a stream.
2. The explicit finite element flow model solves the flood routing problems, having a maximum time step of two seconds.
3. The weighted-implicit finite element model yields accurate results, with a maximum time interval of ten minutes and weighting factor in a range of 0.55 to 1.00.
4. Both the finite element models have been tested against a problem presented by Viessman et al. The results of flood hydrographs are in close agreement.

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PUBLICATIONS

The following publications are currently being prepared:

1. Finite Element Models in Flood Routing, Technical Report, School of Civil Engineering, Oklahoma State University, Stillwater, Oklahoma.

2. Explicit Finite Element Model in Streamflow Routing, Water Resources Research.

3. Flood Routing by Weighted - Implicit Finite Element Method. J. of Hyd. Div., Am. Soc. Civil Engrs.