Functions Reconstructed from Intercepts of Tangents

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Every twice differentiable function of one variable has a unique tangent line at each point of its domain. Each tangent line may or may not have an *x*- and a *y*-intercept. If none of the tangent lines have both intercepts then knowledge of the other intercept makes it a trivial task to reconstruct the original function. For many functions whose tangent lines do possess *x*- and *y*-intercepts, we have outlined a technique by which the original function may be recovered from just the *x*- and *y*-intercepts of the tangent lines. Knowledge of the point of tangency between the tangent lines and the original function is not necessary. Furthermore, twice differentiable functions of two variables have unique tangent lines at each point in their domains. However, the tangent lines to such functions have *xy*-plane, *xz*-plane, and *yz*-plane intercepts instead of the simpler *x*- and *y*- intercepts. Still, our technique for reconstructing functions of one variable can be extended to reconstruct many functions of two variables merely with the knowledge of the location of the planar intercepts of the tangent lines. Additionally, we have developed a technique to reconstruct a surface given only the points where its tangent planes intersect the *x*, *y*, and *z*-axes.

INTRODUCTION

It seems reasonable to be able to physically reconstruct, say, a football if all that was known about it was its set of tangent planes. The points of tangency between the football and the tangent planes would not be necessary. In this paper, we will demonstrate a technique by which most curves in 2-space, most curves in 3-space, and most surfaces in 3-space can all be reconstructed from the set containing only the intercepts of its tangents with the coordinate axes or planes.

RESULTS AND DISCUSSION

Reconstructing Curves in R^2

In this paper, we will limit ourselves to all explicit twice differentiable functions such that no subset thereof is a straight line. If a subset of the function is a straight line such that, when extended, passes through the origin, then this subset of points of the function will have intercepts consisting of only the origin. As there are infinitely many straight lines through the origin, then more information would be needed in order to determine which line it is. If a subset of the function is a straight line such that, when extended, does not pass through the origin, then the intercepts belonging to this subset of the function will remain constant. By the continuity of the function, it should be obvious which subset of the function would belong to the straight line, assuming we were aware of the domain of the function. So, suppose that $f(t) = (x(t), y(t))$ is a twice differentiable function of t such that no subset thereof is a straight line. If no tangent line of f has an x-intercept then $f(t) = (0,b)$, for some constant b. If no tangent line of f has a y-intercept then $f(t) = (a,0)$, for some constant a. If $y'(t)/x'(t) = 0$, then f has a tangent line at t of the form $y = c$, for some constant c. If $x'(t)/y'(t) = 0$, then f has a tangent line at t of the form $x = d$, for some constant d. Suppose that there are at most countably many points t such that $x'(t) = 0$ or $y'(t) = 0$. Then any nonvertical and nonhorizontal tangent line to this function will have the form

$$y - y(t) = \frac{y'(t)}{x'(t)} (x - x(t)).$$

Therefore, the x-intercept of the tangent line will be

$$x - x(t) = \frac{x'(t)}{y'(t)} y(t). \qquad (1)$$

and the y-intercept will be

$$y - y(t) = \frac{y'(t)}{x'(t)} x(t). \qquad (2)$$

It is known that it is possible to reconstruct a function from its set of tangent lines (Horwitz 1989). We show below that if the only information that we possessed about f was the x-intercepts and the y-intercepts of its set of tangent lines then we could still reconstruct the original function f. That is, it is possible to determine what a function is from the set containing the intercepts of its tangent lines. Note that it is not necessary to know the location of the points of tangency between the tangent lines and the original function f.

We begin by describing a tangent line transformation of a function f, as presented in (Butler 2003). If $f(t) = (x(t), y(t))$ is any function of t, then the *tangent line transformation M* is defined by

$$M(x(t), y(t)) = \left[\frac{y'(t)}{x'(t)}, y(t) - \frac{y'(t)}{x'(t)} x(t)\right] \qquad (3)$$

where $\frac{y'(t)}{x'(t)}$ provides the slope of the (nonvertical) line tangent to the curve $f(t) = (x(t), y(t))$ at $x = x(t)$, and $y(t) - \frac{y'(t)}{x'(t)} x(t)$ provides the value of the y-intercept of the tangent line. Consider now the sets A and B, where A = {intercepts of all vertical and horizontal tangent lines}
and

$$B = \left\{\left(x(t) - \frac{x'(t)}{y'(t)} y(t), 0\right), \left(0, y(t) - \frac{y'(t)}{x'(t)} x(t)\right): x'(t), y'(t), \neq 0\right\}$$

$T_1 = A \cup B$ is a set containing all intercepts of the tangent lines to $y = f(x)$. Note that the point of tangency between the tangent line and the function $y = f(x)$ is unknown.

Given the set T_1, it is possible to reconstruct the function f from which T_1 is derived. First, note that if T_1 has only one element, then reconstructing f is trivial. Otherwise, note that the quotient of the y-intercept over the x-intercept provides the negative of the slope of the tangent line at $x = x(t)$. That is, we have

$$\frac{y(t) - \frac{y'(t)}{x'(t)} x(t)}{x(t) - \frac{x'(t)}{y'(t)} y(t)} = \frac{\frac{x'(t)y(t) - y'(t)x(t)}{x'(t)}}{\frac{x(t)y'(t) - x'(t)y(t)}{y'(t)}} = -\frac{y'(t)}{x'(t)}.$$

Now we evaluate

$$M\left[-\frac{y'(t)}{x'(t)}, y(t) - \frac{y'(t)}{x'(t)} x(t)\right].$$

By (3), the first coordinate is

$$\frac{\frac{d}{dt}\left[y(t) - \frac{y'(t)}{x'(t)} x(t)\right]}{\frac{d}{dt}\left[-\frac{y'(t)}{x'(t)}\right]} = \frac{y'(t) - \frac{y''(t)x'(t) - y'(t)x''(t)}{[x'(t)]^2} x(t) - \frac{y'(t)}{x'(t)} x'(t)}{-\frac{y''(t)x'(t) - y'(t)x''(t)}{[x'(t)]^2}} = x(t), \qquad (4)$$

and the second coordinate is

$$\left[y(t) - \frac{y'(t)}{x'(t)} x(t)\right] - \frac{\frac{d}{dt}\left[y(t) - \frac{y'(t)}{x'(t)} x(t)\right]}{\frac{d}{dt}\left[-\frac{y'(t)}{x'(t)}\right]} \left[-\frac{y'(t)}{x'(t)}\right]$$

$$= y(t) - \frac{y'(t)}{x'(t)}\, x(t) - \frac{y'(t) - \dfrac{y''(t)x'(t) - y'(t)x''(t)}{[x'(t)]^2}\, x(t) - \dfrac{y'(t)}{x'(t)}\, x'(t)}{-\dfrac{y'(t)x'(t) - y'(t)x''(t)}{[x'(t)]^2}} \left[-\frac{y'(t)}{x'(t)}\right]$$

$$= y(t) \tag{5}$$

Therefore, from (4) and (5) we see that

$$M\left[\frac{y'(t)}{x'(t)},\, y(t) - \frac{y'(t)}{x'(t)}\, x(t)\right] = (x(t), y(t)). \tag{6}$$

This provides the original function. Thus, the original function is obtained from nothing more than the information found in the set T_1.

Reconstructing Curves in R^3

As in the case for R^2, we again limit ourselves to all explicit twice differentiable functions such that no subset thereof is a straight line. Suppose that $f(t) = (x(t), y(t), z(t))$ is such a curve. Then for any fixed t in its domain the tangent line at t is given by

$$(x(s), y(s), z(s)) = (x(t), y(t), z(t)) + s(x'(t), y'(t), z'(t)). \tag{7}$$

We call the xy-plane, the xz-plane, and the yz-plane the *major planes*. If every tangent line of f possesses planar intercepts belonging only to two of the three major planes, then clearly the problem of reconstructing f reduces to the 2-dimensional case. If the tangent line l in (7) does not possess two planar intercepts, say l does not possess an x_1x_2 plane intercept and an x_1x_3 plane intercept, then the tangent line will have parametric equations of the form

$$x_1(s) = s$$
$$x_2(s) = a$$
$$x_3(s) = b$$

where a and b are nonzero constants. If the tangent line l in (7) does not possess exactly one planar intercept, say l does not possess an x_1x_2 plane intercept, then the tangent line will have parametric equations of the form

$$x_1(s) = a + s \cdot b$$
$$x_2(s) = c + s \cdot d$$
$$x_3(s) = k,$$

where a and c are any constants, and b, d and k are nonzero constants. If the tangent line l in (7) is contained in one of the three planes, say the x_1x_2 plane, then the tangent line will have parametric equations of the form

$$x_1(s) = a + s \cdot b$$
$$x_2(s) = c + s \cdot d$$
$$x_3(s) = 0,$$

where a, b, c, and d are constants and at least one of b or d is nonzero. If the tangent line in (7) possesses all three planar intercepts then the point at which it intersects the xy plane is determined by setting $z(s) = 0$. This gives

$$0 = z(t) + s \cdot z'(t),$$

yielding

$$s = \frac{z(t)}{z'(t)}. \tag{8}$$

By substitution of (8) into (7) we get the remaining two coordinates

$$x = x(t) - \frac{x'(t)}{z'(t)}\, z(t) \quad \text{and} \quad y = y(t) - \frac{y'(t)}{z'(t)}\, z(t)$$

Thus, we have that the xy-plane intercept of the tangent lines to $f(t)$ is

$$\left(x(t) - \frac{x'(t)}{z'(t)}\, z(t),\, y(t) - \frac{y'(t)}{z'(t)}\, z(t),\, 0\right) \tag{9}$$

In a similar manner, we obtain the remaining planar intercepts. We get

xz intercept:

$$\left(x(t) - \frac{x'(t)}{z'(t)}\, y(t),\, 0,\, z(t) - \frac{z'(t)}{y'(t)}\, y(t)\right); \quad (10)$$

yz intercept:

$$\left(0,\, y(t) - \frac{y'(t)}{x'(t)}\, x(t),\, x(t) - \frac{z'(t)}{x'(t)}\, x(t)\right) \quad (11)$$

We will denote the coordinates of these planar intercepts (9), (10), and (11) by

xy intercept = $(x_I, y_I, 0)$,

xz intercept = $(x_{II}, 0.z_{II})$, and

yz intercept = $(0.y_{III}, z_{III})$, respectively.

Consider now the sets A and B where

A = {planar intercepts of all tangent lines that are parallel to one of the three planes}
And

$B = \{(x_I, y_I, 0), (x_{II}, 0, z_{II}), (0, y_{III}, z_{III})\}$.

$T_2 = A \cup B$ is a set containing all of the planar intercepts of the tangent lines to $f(t) = (x(t), y(t), z(t))$. Note that the point of tangency between the tangent line and the function $f(t) = (x(t), y(t), z(t))$ is unknown.

Given the set T_2, it is possible to reconstruct the function f from which T_2 is derived. If one of the coordinates of every intercept in T_2 is the same value, then, as alluded to above, the problem reduces to the 2-dimensional case. Otherwise, we evaluate

$$M\left[\frac{x_I - x_{II}}{z_{II}}, x_1\right], \quad (12)$$

This equals

$$M\left[\frac{\left[x(t) - \frac{x'(t)}{z'(t)} z(t)\right] - \left[x(t) - \frac{x'(t)}{y'(t)} y(t)\right]}{z(t) - \frac{z'(t)}{y'(t)} y(t)}, x(t) - \frac{x'(t)}{z'(t)} z(t)\right]$$

$$= M\left[\frac{\dfrac{x'(t)y(t)z'(t) - x'(t)y'(t)z(t)}{y'(t)z'(t)}}{\dfrac{z(t)y'(t) - z'(t)y(t)}{y'(t)}}, x(t) - \frac{x'(t)}{z'(t)} z(t)\right] = M\left[-\frac{x'(t)}{z'(t)}, x(t) - \frac{x'(t)}{z'(t)} z(t)\right]. \quad (13)$$

As has already been seen, (13) is equal to $(z(t), x(t))$.
That is,

$$M\left[\frac{x_I - x_{II}}{z_{II}}, x_I\right] = (z(t), x(t)).$$

Therefore, two of the coordinates of the original function have been reconstructed. To reconstruct the third coordinate $y(t)$, we evaluate

$$M\left[\frac{y_{II} - y_{III}}{z_{III}}, y_{II}\right].$$

By a similar derivation as above it can be shown that

$$M\left[\frac{y_{II} - y_{III}}{z_{III}}, y_{II}\right] = M\left[-\frac{y'(t)}{z'(t)}, y(t) - \frac{y'(t)}{z'(t} z(t)\right]$$
$$=(z(t), y(t)).$$

and, hence, all three coordinates of the original function $f(t) = (x(t), y(t), z(t))$ have been reconstructed. Thus, the original function is obtained from nothing more than the information found in the set T_2.

Reconstructing Surfaces in R^3

Now, suppose that $z = f(x, y)$ is a differentiable surface whose intercepts consist of something more than only the origin. As there are infinitely many planes that intersect only the origin, then more information would be needed in order to reconstruct it. If the tangent planes of f intersect only one of the coordinate axes, say the x_1 axis, then f will be a plane of the form $x_1 = b$, for some constant b. If the tangent planes of f intersect exactly two of the coordinate axes at points, say the x_1 and x_2 axes, then f will be of the form $f\,(x_1, x_2) = c$, and the problem of reconstructing f reduces to the 2-dimensional case of reconstructing a curve. Otherwise, for any fixed point (x_0, y_0) in its domain the tangent plane at (x_0, y_0) is given by

$$z = f\,(x_0, y_0) + f_{\mathrm{x}}(x_0, y_0)\,(x - x_0) + f_{\mathrm{y}}\,(x_0, y_0)(y - y_0). \tag{14}$$

The point z_1 at which this tangent plane intersects the z-axis is determined by setting $x = 0$ and $y = 0$. Thus, the point x_1 on the z-axis that the tangent plane intersects is

$$z_1 = f\,(x_0, y_0) - x_0 f_{\mathrm{x}}(x_0, y_0) - y_0 f_{\mathrm{y}}\,(x_0, y_0). \tag{15}$$

The point x_1 at which the tangent plane intersects the x-axis is determined by setting $y = 0$ and $z = 0$. This provides

$$0 = f\,(x_0, y_0) + f_{\mathrm{x}}\,(x_0, y_0)(x_1 - x_0) - y_0 f_{\mathrm{y}}\,(x_0, y_0). \tag{16}$$

Rewriting (16) gives the point x_1 on the x-axis that the tangent plane intersects is

$$x_1 = \frac{y_0 f_y(x_0, y_0) + x_0 f_x(x_0, y_0) - f\,(x_0, y_0)}{f_x\,(x_0, y_0)} \tag{17}$$

A similar derivation from (14) will give the point y_1 on the y-axis that the tangent plane intersects which is

$$y_1 = \frac{x_0 f_x(x_0, y_0) + y_0 f_y(x_0, y_0) - f\,(x_0, y_0)}{f_x\,(x_0, y_0)} \tag{18}$$

Consider now the sets

A = {intercepts of tangent planes to f that do not intersect all 3 coordinate axes}

and

$B = \{(x_1, 0, 0), (0, y_1, 0), (0, 0, z_1)\}$.

$T_3 = A \cup B$ is a set containing all of the points where each tangent plane to the surface $y = f\,(x, y)$ intersects each of the three axes. Note that the point of tangency between the tangent plane and the surface is unknown.

Given the set T_3, it is possible to reconstruct the function f from which T_3 is derived. If there is only one element in T_3 then reconstructing f is a trivial matter. If one of the coordinates of every intercept in T_2 is either 0 or the entire axis, then, as alluded to above, the problem reduces to the 2-dimensional case. Otherwise, we claim that

$$f(x,y) = z_1\left(1 - \frac{x}{x_1} - \frac{y}{y_1}\right) \tag{19}$$

To demonstrate this with less complex notation, we will drop the 0 subscripts. Note that

$$z_1\left[1 - \frac{x}{y} - \frac{\quad}{\quad}\right] = (f(x, y) - xf_x(x, y) - yf_y(x, y))\left[1 - \frac{x}{\dfrac{yf_y(x.y) + xf_x(x, y) - f\,(x, y)}{f_x\,(x, y)}} - \frac{y}{\dfrac{xf_x(x,y) + yf_y(x, y) - f\,(x, y)}{f_x\,(x, y)}}\right]$$

$$= f\,(x, y) - xf_x\,(x, y) - yf_y\,(x, y) + xf_x\,(x, y) + yf_y(x, y)$$

$$= f\,(x, y).$$

Hence, the original function is obtained from nothing more than the information found in the set T_3.

Reconstruction in Higher Dimensions

Currently research is underway to determine techniques that can be used to reconstruct curves in R^n for $n \geq 4$. We conjecture that the tangent line transformation will be used to reconstruct curves in R^n for $n \geq 4$. Reconstructing surfaces in these higher dimensional spaces is actually a simple generalization of the technique described for surfaces in R^3. We leave it to the interested reader to work out the details.

REFERENCES

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