# Radiation Damping in the Mechanical Oscillator

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**The typical model of the damped mechanical oscillator is the simple spring, fixed at one end, with a mass attached to the other. Air resistance is assumed to provide a retarding force which is proportional to the object's velocity. The equation of motion is easily solved and the solution is often employed to describe a variety of physical phenomena. In this work, a different model of radiation damping is considered for the spring-mass system. The (primary) spring is considered to possess mass. Air resistance is neglected, but an infinitely long (secondary or radiation) spring is considered to be attached to the mass. Thus, if the mass is set in motion, the infinite spring is excited and energy is carried out of the system. The mechanical wave in the secondary spring is analogous to radiation, and is referred to as mechanical radiation. Thus, while the mathematics of this system is more involved than that of the traditional damped oscillator, the model provides a more realistic description of radiation damping. The damping factor and frequencies are calculated for various choices of system parameters. The results of the calculation are generally consistent with expectation. Damping is clearly exhibited, and the frequencies are lowered by the damping. Damping is more important for the lower modes.**

## INTRODUCTION

The damped mechanical oscillator plays an important role in physics. It is a simple system, is analogous to the LCR circuit, and provides a simple solution, which can model a variety of more complex physical phenomena. Students may encounter the system at the level of calculus-based physics as well as in more advanced courses. Examples of the system are found in textbooks on mechanics (1), wave motion and acoustics (2), electromagnetic theory and optics (3), atomic and solid state physics (4), and other areas as well.

The damped oscillator is usually introduced by means of the following traditional model. A spring, which is considered to be massless and to obey Hooke's law with spring constant $k$ , has one end fixed. To the other end is attached a mass, $M$. A retarding force proportional to the velocity of the mass is also assumed to act. If the damping force is taken to be $-q\mathrm{d}x/\mathrm{d}t$, then, when set in motion, the displacement, $x(t)$, of the mass from equilibrium at time $t$ is described by

$$\mathrm{d}^2x/\mathrm{d}t^2 + 2\,\beta\,\mathrm{d}x/\mathrm{d}t + \omega_0^2 x = 0,$$

where $\beta = q/2M$, and $\omega_0^2 = k/M$. The solution is well known to be

$$x = \mathrm{A}e^{-\beta t}\cos(\omega t + \phi),$$

where $\phi$ is the phase and $\omega$, which is interpreted as the angular frequency, is given by $\omega = (\omega_0^2 - \beta^2)^{1/2}$. This assumes that $\omega_0^2 > \beta^2$, which is, typically, the most important case.

For many applications, the factor $\beta^2$ is small, and so one may write

$$x = \mathrm{A}e^{-\beta t}\cos(\omega_0 t + \phi), \qquad (1)$$

Thus, Eq. 1 describes approximately harmonic oscillations with amplitude which decreases exponentially with time. The frequency of oscillation is approximately equal to that of the undamped oscillator, although, in general, the damping also produces a reduction in the frequency of oscillation.

This treatment of the damped oscillator is very useful, primarily because the equation of motion is easily solved, even if the physics is not necessarily so simple. In a mechanical system, for example, the assumption that air resistance varies directly as velocity may not be strictly valid. In the Lorentz model of the atom (Carlstone, manuscript in preparation), the linear velocity assumption is still employed, although it is radiation which is responsible for the retarding force, and the relation between the radiation and the damping is not so simple. Or in the context of the solid, damping may be a result of collisions between an ion and the surrounding solids, which, in turn, produces acoustic waves, or phonons. The important feature of the models is the exponential damping, as in Eq. 1. There are, of course, other models that give rise to exponential damping.

The purpose of this work is to present a different mechanical model for damping of the oscillator. Rather than consider the damping to be produced by air resistance, a model is presented in which the damping is caused by mechanical waves which carry energy out of the system. Such waves are analogous to radiation losses, and shall therefore be referred to as mechanical radiation. This model is more consistent with the applications mentioned above than is the traditional Lorentz model which is based on an air-damped oscillator.

Fig. 1 represents the system of interest. The medium in which the mechanical energy loss occurs is an infinitely long spring which is attached to the oscillating body. The most general situation is that in which the primary spring, as well as the radiation spring, is considered to possess mass. The resulting equations of motion turn out to be similar to those which have been given for a related mechanical system (5). By considering the mass of the spring to be non-zero, there will be an infinite number of normal modes of oscillation to describe. Of course, only the lowest frequencies are typically of interest. After a review of the wave motion in the mass-loaded spring, these equations are presented and solved. Specific numerical results are then presented and discussed.

## THE MASS-LOADED SPRING

In order to appreciate the effects of radiation damping on the system of interest, it is desirable to first review the properties of the undamped system. As observed above, the mass of the primary spring is taken to be non-zero. Therefore, this section provides a description of the modes of oscillation of the mass-loaded spring for non-negligible spring mass and with no damping.

The equations of motion for the waves in a spring have been described by several authors (6-10). The displacement from equilibrium, $\Xi(x,t)$, at time $t$, of an element of spring which is at the equilibrium position, $x$, satisfies the following equation for a spring with variable density $\mu(x)$:

$$\partial^2\Xi/\partial x^2 - (\mu(x)/kL)\,(\partial^2\Xi/\partial t^2) = 0. \quad (2)$$

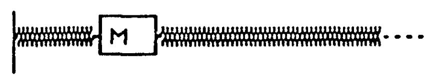


Figure 1. The oscillator with radiation damping

If the spring has uniform density, this reduces to the familiar classical wave equation. For a uniform spring which is loaded with point mass $M$, at $x=L$, the density may be expressed

$$\mu(x) = \mu_o + M\,\delta(x - L),$$

where $\mu_o$ is the constant density of the spring. It is useful to introduce the phase velocity of the wave, $v$, in a uniform spring, which is given by

$$v^{-2} = \mu_o/kL = m/kL^2, \quad (3)$$

where $m$ is the mass of the spring.

The time dependence is separated with the substitution $\Xi(x,t)=\xi(x)\exp(-i\omega t)$, so that

$$\partial^2\xi(x)/\partial x^2 + \omega^2 v^{-2}\,\xi(x) = -(M\omega^2/P)\,\delta(x)\xi(x), \quad (4)$$

with $P=kL$. The presence of the Dirac delta function in Eq. 4 suggests that a convenient solution is

$$\xi(x) = (M\omega^2/P)G(x,L)\xi(L), \quad (5)$$

where $G(x,x')$ is the Green's function for the spring which is fixed at $x = 0$ and free at $x = L$. It is given by

$$G(x,x') = [v/\omega\cos(\omega L/v)] \times \sin(\omega x_</v)\cos[\omega(L - x_>)/v], \quad (6)$$

where $x_<$ ($x_>$) is the lesser (greater) of $(x,x')$. When Eq. 5 is evaluated at $x=L$, the substitution of Eq. 6 yields

$$\alpha\tan(\alpha) = m/M, \quad (7)$$

where, again, $m/M$ is the ratio of the spring mass to the mass load, and $\alpha = \omega L/v$. Eq. 7 is a transcendental equation, and its roots may be found once the ratio $K=M/m$ is specified. Each root corresponds to a different normal mode of oscillation, and is customarily labeled by an index $n$ = 1,2,3,. . .. The angular frequency of each of the $n$th normal mode is then represented as

$$\omega_n = \alpha_n v/L. \quad (8)$$

Several roots, $\alpha_n$, of Eq. 7 have been found for $K$= 1/4, 1, and 4. These roots, reported in Table 1 under "MLS", provide the frequency of the mass-loaded spring (MLS) for the case of no radiation damping; they are to be compared with the other values of $\alpha_n$, which correspond to damped motion with different ratios of the characteristic impedance of the springs and different *K*.

It should be observed that Eq. 7 may also be obtained by choosing the mass load to be external to the system. If that is done, the homogeneous form of Eq. 4 applies and the solution is subject to the boundary conditions $\xi(0)=0$, $(d\xi/dx)_{x=L}=(M\omega/kL)\ \xi(L)$. This is a different way to obtain Eq. 7 and it is the approach taken by the references cited above.

## THE MASS-LOADED SPRING with RADIATION DAMPING

As described above and in Fig. 1, the system of interest is that in which a uniform spring, possessing non-negligible mass, has the end at $x = 0$ fixed. A point mass, *M*, is attached at the other end, $x=L$, and attached to the mass is a semi-infinite Spring. Thus when the mass is set in oscillation, the semi-infinite spring carries energy out of the system (mechanical radiation), and damping of the oscillator occurs.

If a uniform spring is subject to a driving force, or other interaction, per unit length, $F(x,t)$, then the equation of motion is

$$(\partial^2\Xi/\partial x^2) - v^{-2}\ (\partial^2\Xi/\partial t^2) = -F(x,t)/P$$

where, again, $P=kL$ is a convenient simplification; it carries dimensions of force. In cases of interest in this work, the forces arise as the result of one part of the system interacting with the other. Such action-reaction forces will be denoted by the general symbol *N*.

When all of the above considerations are made, the equations of motion for the system may be developed. It is convenient to consider the first part of the system to be the finite spring with (point) mass-load *M* as described earlier. The second part of the system is the infinite spring. Fig. 2 represents the system. When the interaction between the two parts is included, the equations of motion for the two parts of the system are:

$$(\partial^2\Xi_1/\partial x^2) - v_1^{-2}\ (\partial^2\Xi_1/\partial t^2) = (M\ \delta(x-L)/P_1)\ (\partial^2\Xi_1/\partial t^2) - (N_1(x,t)/P_1)\ \delta(x-L),$$

$$(\partial^2\Xi_2/\partial x^2) - v_2^{-2}\ (\partial^2\Xi_2/\partial t^2) = (N_2(x,t)/P_2)\ \delta(x-L)$$

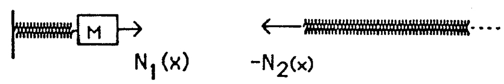


Figure 2. Action–reaction forces between the two main subsystems.

Clearly, the indices 1 and 2 refer to the respective parts of the system. $N_1$ and $-N_2$ represent the action-reaction pair. The presence of the Dirac delta function indicates that the reaction between the two parts of the system occurs at the point $x=L$, and it also insures dimensional consistency. In the case of the infinite spring, $P_2=k_2L_2$, where $L_2$ represents some arbitrary length of spring, and $k_2$ is the corresponding spring constant. (This is discussed in a manuscript in preparation (Carlstone)).

Conditions imposed on the solutions to the above equations are that the displacements be equal at $x=L$, $\Xi_1(x=L,t) = \Xi_2(x=L,t)$, and the action-reaction condition, $N_1(x=L,t) = N_2(x=L,t)$. After elimination of the time dependence according to

$$\Xi_1(x,t) = \xi_1(x)\ \exp(-i\omega t) \qquad (9)$$

$$\Xi_2(x,t) = \xi_2(x)\ \exp(-i\omega t) \qquad (10)$$

including the condition that the reaction will obey the same time dependence,

$$N_1(x,t) = N_1(x)\ \exp(-i\omega t)$$

$$N_2(x,t) = N_2(x)\ \exp(-i\omega t),$$

the problem reduces to that of solving

$$\partial^2\xi_1/\partial x^2 + \omega^2 v_1^{-2}\ \xi_1 = -(M\omega^2/P_1)\delta(x-L)\ \xi_1 - (N_1(x)/P_1)\ \delta(x-L) \qquad (11)$$

$$\partial^2\xi_2/\partial x^2 + v_2^{-2}\ \xi_2 = (N_2(x)/P)\ \delta(x-L) \qquad (12)$$

with $\xi_1(L) = \xi_2(L)$, and $N_1(L) = N_2(L)$. The presence of the delta function in Eqs. 11 and 12 again suggests that a convenient solution may be given in terms of the Green's functions for the respective systems:

$$\xi_1(x) = (M\omega^2/P_1)\xi_1(L)G_1(x,L) + (N_1(L)/P_1)G_1(x,L) \qquad (13)$$

$$\xi_2(x) = -\ (N_2(L)/P_2)G_2(x,L). \qquad (14)$$

Here $G_1(x,x')$ is the Green's function for the spring which is fixed at $x = 0$ and free at $x=L$. This is of the same form as Eq. 6, but in terms of the notation of this section,

TABLE 1. Frequencies and damping coefficients for the normal modes.

| | | MLS | R=1/4 | | R=1 | | R=4 | |
|---|---|---|---|---|---|---|---|---|
| K | n | α | α | β | α | β | α | β |
| 1/4 | 1 | 1.266 | 1.255 | 0.181 | 1.050 | 0.841 | 0 | 0.259 |
| | 2 | 3.935 | 3.924 | 0.112 | 3.751 | 0.404 | 3.194 | 0.248 |
| | 3 | 6.814 | 6.808 | 0.060 | 6.728 | 0.212 | 6.375 | 0.22 |
| | 4 | 9.812 | 9.809 | 0.034 | 9.770 | 0.125 | 9.540 | 0.185 |
| | 5 | 12.868 | 12.866 | 0.021 | 12.847 | 0.080 | 12.691 | 0.152 |
| 1 | 1 | 0.860 | 0.856 | 0.091 | 0.782 | 0.370 | 0 | 0.275 |
| | 2 | 3.425 | 3.424 | 0.018 | 3.410 | 0.069 | 3.272 | 0.149 |
| | 3 | 6.437 | 6.437 | 0.006 | 6.434 | 0.023 | 6.400 | 0.069 |
| | 4 | 9.529 | 9.529 | 0.003 | 9.528 | 0.011 | 9.514 | 0.037 |
| | 5 | 12.645 | 12.645 | 0.002 | 12.645 | 0.006 | 12.638 | 0.023 |
| 4 | 1 | 0.480 | 0.479 | 0.029 | 0.466 | 0.115 | 0.462 | 0.133 |
| | 2 | 3.219 | 3.219 | 0.001 | 3.219 | 0.006 | 3.213 | 0.022 |
| | 3 | 6.323 | 6.322 | 0.0004 | 6.323 | 0.002 | 6.322 | 0.006 |
| | 4 | 9.451 | 9.451 | 0.0002 | 9.451 | 0.0007 | 9.451 | 0.003 |
| | 5 | 12.586 | 12.586 | 0.0001 | 12.586 | 0.0004 | 12.586 | 0.002 |

$$G_1(x,x') = (v_1/\omega \cos(\omega L/v_1)) \times \sin(\omega x_</v_1) \cos[\omega(L - x_>)/v_1],$$

where $x_<$ ($x_>$) is the lesser (greater) of $(x,x')$. For the semi-infinite spring which is free at $x = L$ the function is:

$$G_2(x,x') = (i v_2/\omega)\cos[\omega(x_< - L)/v_2] \times \exp[\omega(x_> - L)/v_2],$$

If Eqs. 13 and 14 are evaluated at $x = L$, the imposition of the boundary conditions and substitution of the appropriate Green's functions yield the following:

$$(v_1/P_1\omega)\tan(\omega L/v_1) + (iv_2/P_2\omega) \times [1-(Mv_1^2/P_1L)(\omega L/v_1)\tan(\omega L/v_1)] = 0$$

This is reduced to the form

$$R\tan(\gamma) + i[1-K\gamma\tan(\gamma)] = 0, \qquad (15)$$

by the substitutions $R = (P_2/v_2)/(P_1/v_1)$, $K = M v_1^2/(P_1 L) = M/m$, and $\gamma = \omega L/v_1$. $R$ may be interpreted as the ratio of the characteristic mechanical impedance of spring 2 to that of spring 1 (Carlstone, manuscript in preparation). Small $R$ corresponds to a very weak "radiation" field, or to weak interaction collisions with the surrounding. $K$ is seen to be the ratio of the mass load to the mass of the spring. And $\gamma$ is determined by the adjustable parameters $R$ and $K$. From Eq. 15 it is clear that $\gamma$ is complex.

The substitution $\gamma = \alpha - i\beta$ yields the anticipated form for Eqs. 9 and 10,

$$\Xi_1(x,t) = \xi_1(x)\exp(-\beta v_1 t/L) \times \exp(-i\alpha v_1 t/L) \quad (16) \qquad \Xi_2(x,t) = \xi_2(x)\exp(-\beta v_1 t/L) \times \exp(-i\alpha v_1 t/L),$$

and also allows Eq. 15 to be separated into real and imaginary parts, the result of which is

$$(R-\beta K)\sin(\alpha)\cosh(\beta) = \sinh(\beta)[K\alpha\cos(\alpha) + \sin(\alpha)] \quad (17)$$

$$(R-\beta K)\cos(\alpha)\sinh(\beta) = \cosh(\beta)[-K\alpha\sin(\alpha) + \cos(\alpha)] \quad (18)$$

These are, of course, coupled transcendental equations. There are an infinite number of paired roots, $\alpha$ and $\beta$, each pair corresponding to a normal mode of oscillation. The real part of $\omega$ provides the angular frequency, and the angular frequency is related to $\alpha$ as given in Eq. 8. Unfortunately, perhaps, it is not possible to obtain closed form expressions for $\alpha$ and $\beta$ in terms of the parameters of the system, $R$ and $K$. Once values are selected for these parameters, the coupled transcendental equations may be easily solved. Mathematica™ has been employed, and the results for $R$ and $K$ = 4, 1, and 1/4 are given in Table 1.

The displacement of the mass, $X_n(t)$, when in the $n$th mode of oscillation, is obtained from Eq. 16 when evaluated at $x=L$:

$$X_n(t) = \Xi_{1n}(L,t) = \xi_n(L)\exp(-\beta_n v_1 t/L)\exp(-i\alpha_n v_1 t/L$$

The real part of this yields, as expected, the same form as Eq. 1.

## DISCUSSION

The results in Table 1 are generally consistent with expectation. First, it is observed that Eq. 16 and Eq. 17 are invariant with respect to the substitution $\alpha \rightarrow -\alpha$.

Thus, for every positive solution for $\alpha$ there is a corresponding negative solution. The, negative values have no physical meaning and are not reported. The values which have been found for $\beta$ are all positive, which is consistent with the interpretation that $\beta$ represents a damping coefficient. For $K$=1/4, the greatest values of $\beta$ occur for $R$ = 1, corresponding to the case in which the characteristic impedance of the two springs are equal. In this case the mass-load is small, and impedance matching of the two springs produces the greatest radiation. For higher values of $K$, the impedance of the mass-load becomes important, and there is variation in the behavior of $\beta$ from mode to mode.

It is also seen that the frequencies of the normal modes of oscillation of the system are lowered by the radiation. As in the traditional damped oscillator, this is anticipated. The effect is more significant for the lower frequencies. One other feature stands out: For $R$ = 4 and $K$ = 1/4 and 1, the solution for $\alpha$ is consistent with zero. This is analogous to overdamped motion of the simple oscillator.

The model which is presented here is obviously more complex than the traditional damped oscillator model which serves so well. The advantage of the present model is that it may allow better visualization of radiation damping, as well as visualization of the interaction between the oscillator and the "radiation field".

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