

# Application of Differential Quadrature to Solution of Pool Boiling in Cavities

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**The pool boiling problem in cavities is analyzed from the point of heat transfer by dividing the solution domain into three rectangular regions which are connected through the interface boundary conditions in the form of the compatibility and the continuity relationships. The numerical solution is accomplished by using the method of differential quadrature.**

## INTRODUCTION

Common engineering problems frequently deal with shapes of physical domains having irregular geometries as contrasted to regular ones which consist of single, uniform shapes, such as rectangular, cylindrical or spherical. In general, regular domains in which the physical properties are constant are amenable to analytical solutions. On the other hand, analytical solutions to irregular geometries are not available except in those cases where the irregular domain can be divided into regular shapes such that the system properties are constant within these regions, which are connected through the boundaries along which the system variables are established by the compatibility and the continuity relationships. Several recent examples of these applications are reports by : Hsieh and Su (8), who used a superposition technique along with separation of variables for a cavity geometry; Yi-Zhou and Yi-Heng (11), who used a harmonic function continuation technique on the torsion problems for bars with irregular cross-sections; and Heggs et al. (7), who analyzed a fin assembly heat transfer problem by separation of variables. All these examples were restricted to time-independent or steady-state cases.

However, for problems involving transient-state or steady-state cases with variable system properties, numerical methods such as finite differences, finite elements, or quadrature are unavoidable.

The purpose of this paper is to demonstrate a general approach for analyzing irregular geometries in either time-dependent or time-independent systems. This approach, which requires only a few discrete points to produce numerical solutions, is based on the method of quadrature that was originally proposed by Bellman et al. (1,2) as a technique for rapid solutions of one- and two-dimensional, initial value problems. Subsequently, Mingle (9,10) solved a one-dimensional initial and boundary value problem. Civan and Sliepcevich (3,4,5) extended the technique for multidimensional, steady-state and transient-state boundary value problems, the models for which may involve partial derivatives and integrals.

Essentially the method of differential quadrature replaces a partial space derivative by a linear weighted sum of the function values at the discrete points. As a result, a partial differential equation reduces to a set of algebraic equations if it is time-independent and a set of ordinary differential equations for a time-dependent case. Numerical solution methods for both cases are well developed.

In the present paper the application of the method of quadrature will be demonstrated only on a problem involving an irregular geometry.

## MATHEMATICAL FORMULATION

For the purposes of the present study consider a transient-state conduction problem in

an irregular, two-dimensional geometry for a medium whose properties are dependent only on its temperature. For example, consider the pool boiling of a cryogen in a cylindrical cavity drilled through a solid block, shown in Fig. 1, whose external surfaces are kept at a constant temperature. The heat transfer is represented by the following model:

$$\rho c(\partial T/\partial t) = k[(1/X)(\partial/\partial X)(X\partial T/\partial X) + (\partial^2 T/\partial Y^2)] \quad (1) \quad \text{subject to the initial condition } T = T_o, t = 0 \quad \text{everywhere} \quad (2)$$

and the boundary conditions in the clockwise direction

$$T = T_o, Y = 0, L_2 \leq X \leq (L_1 + L_2) \quad (3)$$

$$T = T_o, X = L_1 + L_2, 0 \leq Y \leq (H_1 + H_2) \quad (4)$$

$$-k\partial T/\partial Y = h(T_L - T), Y = H_1, 0 \leq X \leq L_2 \quad (7)$$

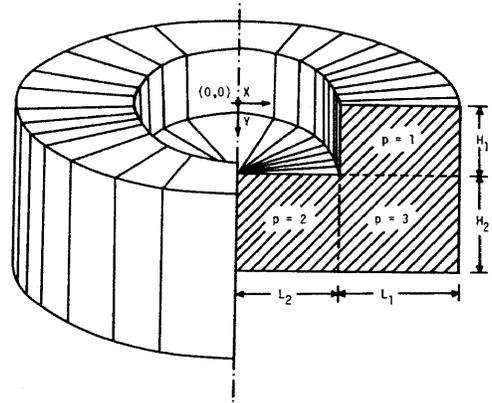
$$T = T_o, Y = H_1 + H_2, 0 \leq X \leq (L_1 + L_2) \quad (5)$$

$$-k\partial T/\partial X = 0, X = 0, H_1 \leq Y \leq (H_1 + H_2) \quad (6)$$

$$-k\partial T/\partial X = h(T_L - T), X = L_2, 0 \leq Y \leq H_1 \quad (8)$$

In Equations 1-8:

- $T$  = temperature
- $T_o$  = outer surface temperature
- $T_L$  = boiling cryogen temperature
- $t$  = time
- $X$  = radial distance
- $Y$  = vertical distance
- $\rho$  = density
- $c$  = specific heat capacity
- $k$  = thermal conductivity
- $h$  = film heat transfer coefficient
- $L_1, L_2, H_1, H_2$  = characteristic lengths and heights for the block shown in Fig. 1



For computational convenience the geometrical domain of interest is separated into three rectangular domains (designated by the letter  $p$ ) through the dashed lines as shown in Fig. 1. Next, each rectangular domain is reduced to a unit square as shown in Fig. 2 by defining the dimensionless distances in  $X$ - and  $Y$ -directions as, for the  $p^{th}$  rectangular domain,

$$x_p = (X - x_{p_o})/L_p \quad (9)$$

$$y_p = (Y - y_{p_o})/H_p \quad (10)$$

in which  $X$  and  $Y$  are measured from the point  $0(0,0)$  shown in Fig. 1,  $L_p$  and  $H_p$  are the width and height, and  $(x_{p_o}, y_{p_o})$  is the origin of the coordinates of the region  $p$ .

Define dimensionless temperature as

$$u = \int_{T_o}^T k(T')dT' / \int_{T_o}^{T_L} k(T')dT'; \quad 0 \leq u \leq 1 \quad (11)$$

Define dimensionless time as

$$\tau = t (k/\rho c)_{\max} / L_{\max}^2 \quad (12)$$

where

$$L_{\max} = \text{maximum of } (L_p; p = 1,2) \quad (13)$$

and

$$(k/\rho c)_{\max} = \text{max of } (k/\rho c) \text{ in } T_L \leq T \leq T_o \quad (14)$$

Thus, by using Equations 9 through 12, Equations 1 through 8 are normalized. The unsteady-state conduction equation becomes

Figure 1. The computational domain and its boundary conditions.

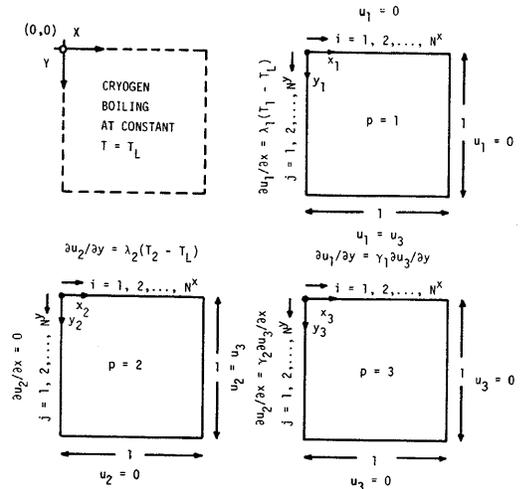


Figure 2. The computational domain normalized with respect to the dimensions of individual regions.

## Solution of the Problem Using Differential Quadrature

$$\partial u_p / \partial \tau = \varphi(T) \alpha_p^2 [(x_p + x_{p_0} / L_p)^{-1} \partial u_p / \partial x_p + \partial^2 u_p / \partial x_p^2 + \beta_p^2 \partial^2 u_p / \partial y_p^2]; \quad p = 1, 2, 3 \quad (15)$$

where

$$\varphi(T) = (k/Qc)_T / (k/Qc)_{\max} \quad (16)$$

$$\alpha_p = L_{\max} / L_p \quad (17)$$

and the so-called aspect ratio

$$\beta_p = (L / H)_p. \quad (18)$$

The initial condition is reduced to

$$u(x_p, y_p) = 0, \quad \tau = 0; \quad p = 1, 2, 3 \quad (19)$$

Boundary conditions for the unit square domain  $p = 1$  in the clockwise direction:

$$u_1 = 0, \quad 0 \leq x_1 \leq 1, \quad y_1 = 0 \quad (20)$$

$$u_1 = 0, \quad x_1 = 1, \quad 0 \leq y_1 \leq 1 \quad (21)$$

The continuity and the compatibility conditions along the interface of domains 1 and 3 are, respectively,

$$u_1 = u_3 \quad (22a)$$

and

$$(\partial u / \partial y)_1 = \gamma_1 (\partial u / \partial y)_3, \quad 0 \leq x_1 \leq 1, \quad y_1 = 1 \quad (22b)$$

in which

$$\gamma_1 = H_1 / H_2. \quad (23)$$

The convective boundary is

$$(\partial u / \partial x_1) = \lambda_1 (T_1 - T_L), \quad x_1 = 0, \quad 0 \leq y_1 \leq 1 \quad (24)$$

where

$$\lambda_1 = L_1 h / \int_{T_0}^{T_1} k(T') dT' \quad (25)$$

Boundary conditions for the unit square domain  $p = 3$  in the clockwise direction:

Equations 22a,b hold for:  $0 \leq x_3 \leq 1, \quad y_3 = 0$ .

$$u_3 = 0, \quad x_3 = 1, \quad 0 \leq y_3 \leq 1 \quad (26)$$

$$u_3 = 0, \quad 0 \leq x_3 \leq 1, \quad y_3 = 1. \quad (27)$$

The continuity and the compatibility conditions between the domains 3 and 2 are

$$u_3 = u_2 \quad (28a)$$

and

$$(\partial u / \partial x)_2 = \gamma_2 (\partial u / \partial x)_3, \quad x_3 = 0, \quad 0 \leq y_3 \leq 1 \quad (28b)$$

in which

$$\gamma_2 = L_2 / L_1. \quad (29)$$

Boundary conditions for the unit square domain  $p = 2$  in the clockwise direction:

$$(\partial u / \partial y)_2 = \lambda_2 (T_2 - T_L), \quad 0 \leq x_2 \leq 1, \quad y_2 = 0 \quad (30)$$

where

$$\lambda_2 = H_2 h / \int_{T_0}^{T_2} k(T') dT'. \quad (31)$$

Equations 28a,b hold for  $x_2 = 1$  and  $0 \leq y_2 \leq 1$ .

$$u_2 = 0, \quad 0 \leq x_2 \leq 1, \quad y_2 = 1 \quad (32)$$

$$(\partial u / \partial x)_2 = 0, \quad x_2 = 0, \quad 0 \leq y_2 \leq 1 \quad (33)$$

The quadrature method replaces a linear operation, such as an integration or a differentiation, on a function by a linear weighted sum of the function values at discrete sample points (2). Therefore, it can be used to obtain numerical solutions. For example, consider

$$du/dx |_{x=x_i} \approx \sum_{j=1}^N w_{ij} u_j \quad (34)$$

in which  $x_i$  are the sample points obtained by decomposing the independent variable range into  $N$  discrete points,  $u_i$  are the function values at these points, and  $w_{ij}$  are the weighting coefficients attached to these points. To calculate the weighting coefficients consider a test function

$$u_k = x^{k-1}; \quad k = 1, 2, \dots, N \quad (35)$$

so that

$$du_k/dx = (k-1) x^{k-2} \quad (36)$$

Substituting Eqs. 35 and 36 into Eq. 34 gives

$$(k-1) x_i^{k-2} = \sum_{j=1}^N w_{ij} x_i^{k-1}; \quad i, k = 1, 2, \dots, N \quad (37)$$

Equation 37 is solved for the weighting coefficients,  $w_{ij}$ , as described by Civan and Sliepcevich (3,4,5). These coefficients are then used in Eq. 34 to approximate the derivative at a discrete point in terms of the function values at all of the discrete points. Weighting coefficients for other types of linear operations can be obtained similarly.

In the following the weighting coefficients associated with the first and second order space derivatives with respect to  $x$  or  $y$  variables will be designated by superscripts in the form of a single and a double  $x$  or  $y$ , respectively.

When the partial space derivatives are replaced by formulae given by Civan and Sliepcevich elsewhere (5), Equations 15 through 33 reduce to the following equations, respectively.

### Transient heat conduction equation:

$$(du/d\tau)_{p,ij} \approx \varphi(T_{p,ij}) \alpha_p^2 \{ [x_{p_i} + (x_{p_0} / L_p)]^{-1} \cdot \sum_{k=1}^{N^x} w_{ik}^x u_{p,kj} + \sum_{k=1}^{N^x} w_{ik}^{xx} u_{p,kj} + \beta_p^2 \sum_{k=1}^{N^y} w_{ik}^{yy} u_{p,ik} \} \\ p = 1, 2, 3; \quad i = 2, 3, \dots, (N^x - 1); \\ j = 2, 3, \dots, (N^y - 1) \quad (38)$$

where

$$\varphi(T_{p,ij}) = (k/\rho c)_{T_{p,ij}} / (k/\rho c)_{\max} \quad (39)$$

$$\alpha_p = L_{\max} / L_p \quad (40)$$

$$\beta_p = (L / H)_p \quad (41)$$

Initial Condition:

$$u_{p,ij} = 0, \quad \tau = 0; \quad p = 1, 2, 3; \\ i = 1, 2, \dots, N^x, \quad j = 1, 2, \dots, N^y \quad (42)$$

Boundary values:

$$u_{1,i1} = 0; \quad i = 1, 2, \dots, N^x \quad (43)$$

$$u_{1,N^y} = 0; \quad j = 2, 3, \dots, (N^y - 1) \quad (44)$$

$$u_{1,iN^y} = u_{3,i1} \quad (45a)$$

$$\sum_{k=1}^{N^y} w_{N^y k}^y u_{1,ik} = \gamma_1 \sum_{k=1}^{N^y} w_{1k}^y u_{3,ik} \quad (45b)$$

where

$$\gamma_1 = H_1 / H_2 \quad (46)$$

$$\sum_{k=1}^{N^x} w_{1k}^x u_{1,kj} = \lambda_1 (u_{1,1j} - 1); \quad j = 1, 2, \dots, N^y \quad (47a)$$

if  $k$  and  $h$  are constant,

where

$$\lambda_1 = L_1 h / k = \text{a constant}, \quad (48a)$$

but

$$\sum_{k=1}^{N^x} w_{1k}^x u_{1,kj} = \lambda_{1,1j} (T_{1,1j} - T_L); \\ j = 1, 2, \dots, N^y \quad (47b)$$

if  $k$  and  $h$  are functions of  $T$ ,

where

$$\lambda_{1,1j} = [L_1 / \int_{T_0}^{T_L} k(T') dT'] h(T_{1,1j}) \quad (48b)$$

$$u_{3,N^y} = 0; \quad j = 1, 2, \dots, N^y \quad (49)$$

$$u_{3,iN^y} = 0; \quad i = 1, 2, \dots, N^x \quad (50)$$

$$u_{2,N^y} = u_{3,1j} \quad (51a)$$

$$\sum_{k=1}^{N^x} w_{N^x k}^x u_{2,kj} = \gamma_2 \sum_{k=1}^{N^x} w_{1k}^x u_{3,kj}; \\ j = 2, 3, \dots, (N^y - 1) \quad (51b)$$

in which

$$\gamma_2 = L_2 / L_1 \quad (52)$$

$$\sum_{k=1}^{N^y} w_{1k}^y u_{2,ik} = \lambda_2 (u_{2,i1} - 1); \quad i = 1, 2, \dots, N^x \quad (53a)$$

if  $k$  and  $h$  are constant

$$\lambda_2 = H_2 h / k = \text{a constant} \quad (54a)$$

$$\sum_{k=1}^{N^y} w_{1k}^y u_{2,ik} = \lambda_{2,i1} (T_{2,i1} - T_L); \quad i = 1, 2, \dots, N^x \quad (53b)$$

if  $k$  and  $h$  are functions of  $T$

$$\lambda_{2,i1} = [H_2 / \int_{T_0}^{T_L} k(T') dT'] h(T_{2,i1}) \quad (54b)$$

$$u_{2,iN^y} = 0; \quad i = 1, 2, \dots, N^x \quad (55)$$

$$\sum_{k=1}^{N^x} w_{1k}^x u_{2,kj} = 0; \quad j = 1, 2, \dots, N^y \quad (56)$$

Equations 38 through 56 given above can be employed together to obtain solution of the actual model at the prescribed discrete points as a function of time. However, before proceeding any further, we express the unknown boundary values, for which derivative boundary conditions are prescribed, and the continuity and the compatibility conditions in terms of the interior domain function values.

First we consider the continuity condition, Eqs. 45a,b. Substituting the known boundary values,  $u_{1,i1} = 0$  and  $u_{3,iN^y} = 0$ , into Eq. 45b and solving Eqs. 45a,b simultaneously give

$$u_{1,iN^y} = u_{3,i1} = \\ [\gamma_1 \sum_{k=2}^{N^y-1} w_{1k}^y u_{3,ik} - \sum_{k=2}^{N^y-1} w_{N^y k}^y u_{1,ik}] / [w_{N^y N^y}^y - \gamma_1 w_{11}^y]; \\ i = 2, 3, \dots, (N^x - 1) \quad (57)$$

By substituting  $u_{1,N^y} = 0$  and arranging, Eq. 47a becomes

$$u_{1,1j} = (\lambda_1 + \sum_{k=2}^{N^x-1} w_{1k}^x u_{1,kj}) / (w_{11}^x - \lambda_1); \\ j = 2, 3, \dots, (N^y - 1) \quad (58a)$$

where

$$\lambda_1 = L_1 h / k = \text{a constant} \quad (59)$$

If  $k$  and  $h$  are functions of temperature, then substituting  $u_{1,N^y} = 0$  and combining Eqs. 47b and 48b give

$$w_{11}^x u_{1,1j} - [L_1 / \int_{T_0}^{T_L} k(T') dT'] h(T_{1,1j}) (T_{1,1j} - T_L) \\ = - \sum_{k=2}^{N^x-1} w_{1k}^x u_{1,kj}; \\ j = 2, 3, \dots, (N^y - 1) \quad (58b)$$

This equation has to be solved for  $u_{1,1j}$  simultaneously with Eq. 11, which relates  $T$  to  $u$ , by a trial-and-error method.

Substituting  $u_{3,N^y} = 0$  and  $u_{3,1j} = u_{2,N^y}$  (Eq. 51a) into Eq. 51b and rearranging give

$$w_{N^x 1}^x u_{2,1j} + (w_{N^x N^x}^x - \gamma_2 w_{11}^x) u_{2,N^y} \\ = \gamma_2 \sum_{k=2}^{N^x-1} w_{1k}^x u_{3,kj} - \sum_{k=2}^{N^x-1} w_{N^x k}^x u_{2,kj} \quad (60)$$

On the other hand Eq. 56 can be arranged as

$$w_{11}^x u_{2,1j} + w_{1,N^x}^x u_{2,N^y} = - \sum_{k=2}^{N^x-1} w_{1k}^x u_{2,kj} \quad (61)$$

Equations 60 and 61 can be solved simultaneously to obtain expressions for  $u_{2,1j}$  and  $u_{2,N^y}$  ( $= u_{3,1j}$ );  $j = 2, 3, \dots, (N^y - 1)$ .

Substituting  $u_{2,iN^y} = 0$  and rearranging Eq. 53a gives for constant  $\lambda_2$  value,

$$u_{2,i1} = (\lambda_2 + \sum_{k=2}^{N^y-1} w_{1k}^y u_{2,ik}) / (w_{11}^y - \lambda_2); \\ i = 2, 3, \dots, N^x \quad (62)$$



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## SYMBOLISM AND NOMENCLATURE

$c$	= specific heat capacity
$d$	= ordinary differentiation operator
$h$	= film heat transfer coefficient
$H$	= characteristic height
$k$	= thermal conductivity
$L$	= characteristic length
$t$	= time
$T$	= temperature
$u$	= dimensionless temperature
$x$	= dimensionless radial distance
$X$	= radial distance
$y$	= dimensionless vertical distance
$Y$	= vertical distance
$w$	= weighting coefficients
$\partial$	= partial differentiation operator
$\int$	= integration operator

### Greek Symbols

$\alpha$	= $L_{\max}/L_p$
$\beta$	= aspect ratio
$\gamma$	= a constant
$\lambda$	= constant
$\rho$	= density
$\tau$	= a dimensionless time
$\varphi$	= diffusivity ratio
$\Sigma$	= summation convention

### Subscripts

$i, j$	= indices indicating the location of a discrete point
$L$	= boiling cryogen
max	= maximum
$N$	= number of discrete points
$o$	= outer surface
$p$	= region index

### Superscripts

$x$	= associated with the first-order partial derivatives in the x-direction
$xx$	= associated with the second-order partial derivatives in the x-direction
$y$	= associated with the first-order partial derivatives in the y-direction
$yy$	= associated with the second-order partial derivatives in the y-direction