Distance-Preserving Maps on Abelian Lattice Ordered Groups

Wayne B. Powell

Department of Mathematics, Oklahoma State University, Stillwater, OK 74078

It is shown that distance-preserving maps defined on an abelian lattice ordered group determine the cardinal summands, and conversely. Those distance preserving maps defined on sublattices of the abelian lattice ordered group that can be extended to the whole group are characterized. Also, it is shown that an abelian lattice ordered group has the property that all such maps are extendable to the whole group if and only if it is strongly projectable.

INTRODUCTION

The order structure on an abelian lattice ordered group allows for the definition of a "distance" function in a natural way. For an element g in an abelian lattice ordered group G it is standard to let $|g| = g \vee (-g)$. We can then define a function $\delta: G \times G \to G$ by $\delta(g_1, g_2) = |g_1 - g_2|$. Swamy (6) calls such a distance function a metric, and the lattice ordered group together with the metric is labelled an autometrized space. He then discusses the geometry of these autometrized spaces.

An important notion in the study of structures with distance functions is that of distance-preserving maps. Let G be a lattice ordered group and L a sublattice of G containing 0. A map $f: L \to G$ is called a congruence if $|f(x) - f(y)| = |x - y|$ for all $x, y \in L$. An isometry is a congruence from G to G. In (7) Swamy shows that every isometry f can be written in the form $f(x) = T(x) + y$ where $y \in G$ and $T: G \to G$ is an isometry, a group automorphism, and an involution (i.e., $TT = T$).

In this note we show that isometries are directly related to the structure of the lattice ordered group. More specifically, the isometries can be completely determined by the cardinal summands. Furthermore, we consider the question of when congruences can be extended to isometries. A condition is given which is necessary and sufficient for an arbitrary congruence to be extended to an isometry, and it is then shown that all

congruences on an abelian lattice ordered group G can be extended to isometries if and only if G is strongly projectable.

Recall that G is a cardinal sum of A and B (written $G = A * B$) if and only if G is a direct sum of A and B where A and B are convex l-subgroups of G. Hence, an element (a,b) of G is positive if and only if both a and b are positive. For each $x \in G$ let $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. Hence, $|x| = x \vee (-x) = x^+ + x^-$ and $x = x^+ - x^-$. If $X \subseteq G$ then the positive cone of X is $X^+ = \{x \in X \mid x \geq 0\}$ and the polar of X is $X^\perp = \{y \in G \mid |x| \wedge |y| = 0 \text{ for all } x \in X\}$. An abelian lattice ordered group is said to be strongly projectable if every polar is a cardinal summand. For other basic properties of lattice ordered groups see Bigard et al. (2) or Fuchs (4).

DETERMINING ISOMETRIES

Throughout this paper G will denote an abelian lattice ordered group. Every such group can be written as a subdirect product of totally ordered groups G_α, $\alpha \in A$. We will denote the αth component of $x \in G \subseteq \prod_{\alpha \in A} G_\alpha$ by x_α.

The following lemma and theorem are due to Swamy [7].

LEMMA 1. Let $L \subseteq G$ be a sublattice containing 0 and let $f: L \to G$ be a congruence. Given $a \in L$ and $\alpha \in A$, either $f(x)_\alpha = -a_\alpha + f(a)_\alpha + x_\alpha$ for all $x \in L$ or $f(x)_\alpha = a_\alpha + f(a)_\alpha - x_\alpha$ for all $x \in L$.

THEOREM 2. If $f:G \to G$ is an isometry of the abelian lattice ordered group G, then there exists a unique involutory, isometric group automorphism T of G such that $f(x) = T(x) + f(0)$ for all $x \in G$.

Theorem 2 says that if we can determine all involutions that are isometries and automorphisms of the group G, then we can determine all isometries of G. The next two theorems show that these involutions are directly related to the structure of the lattice ordered group.

THEOREM 3. Let G be an abelian lattice ordered group. For an involutory, isometric group automorphism T of G set $A = \{T(x)^+ \mid x \in G^+\}$, $B = \{T(x)^- \mid x \in G^+\}$, $\bar{A} = A - A$, and $\bar{B} = B - B$. Then $G = \bar{A} * \bar{B}$.

PROOF. Let T be an involutory, isometric group automorphism of G. Since T is an isometry $|T(x)|_\alpha = |x|_\alpha$ for all $x \in G$, $\alpha \in A$. Hence, $T(x)_\alpha = \pm x_\alpha$ since each G_α is totally ordered. If $x_\alpha \neq 0$ and $T(x)_\alpha = x_\alpha$ for some $x \in G$ and fixed $\alpha \in A$, then $|y_\alpha - x_\alpha| = |T(y)_\alpha - T(x)_\alpha| = |T(y)_\alpha - x_\alpha|$. But then either $y_\alpha = T(y)_\alpha$ or $2x_\alpha - y_\alpha = T(y)_\alpha$. The second of these is impossible since $x_\alpha \neq 0$, so $T(y)_\alpha = y_\alpha$ for all $y \in G$. If $T(x)_\beta = -x_\beta$ for some $x \in G$, $\beta \in A$, then we must have $T(y)_\beta = -y_\beta$ for all $y \in G$. Now let $A = \{T(x)^+ \mid x \in G^+\}$. If $x, y \in G^+$ and $\alpha \in A$, then $T(x + y)_\alpha = \pm(x + y)_\alpha$ so $T(x + y)_\alpha = x_\alpha + y_\alpha$ or $T(x + y)_\alpha = -x_\alpha - y_\alpha$. Thus, $T(x + y)^+ = T(x)^+ + T(y)^+$ and we have $A + A \subseteq A$. If $T(x)^+ \geq y \geq 0$ in G, then $y_\alpha \geq 0$ implies $T(x)_\alpha \geq 0$ so $T(y)_\alpha \neq -y_\alpha$. Hence, $T(y)_\alpha = y_\alpha$ for all $\alpha \in A$ and we have $T(y)^+ = y \in G^+$ making A convex. Set $\bar{A} = A - A$, and $\bar{A}$ becomes a convex l-subgroup of G. Let $B = \{T(x)^- \mid x \in G^+\}$ and $\bar{B} = B - B$. As was the case for $\bar{A}$, we get $\bar{B}$ to be a convex l-subgroup of G. Furthermore, $\bar{A} \cap \bar{B} = \{0\}$, and if $y \in G^+$ then $y_\alpha = T(y)_\alpha = T(y)^+_\alpha$ or $y_\alpha = -T(y)_\alpha = T(y)^-_\alpha$ so $y = T(y)^+ + T(y)^- \in \bar{A} + \bar{B}$. Thus, $G = \bar{A} * \bar{B}$.

Thus, if the isometries are known, then the cardinal summands can also be calculated. For if $f:G \to G$ is an isometry, then

the map $T:G \to G$ defined by $T(x) = f(x) - f(0)$ is an isometric, involutory group automorphism.

Conversely, if the cardinal summands of an abelian lattice ordered group are known, then all isometries can be given.

THEOREM 4. Let G be an abelian lattice ordered group and suppose $G = \bar{A} * \bar{B}$. Then there exists an involutory, isometric group automorphism T of G such that $\bar{A}$ is the subgroup generated by $\{T(x)^+ \mid x \in G^+\}$ and $\bar{B}$ the subgroup generated by $\{T(x)^- \mid x \in G^+\}$.

PROOF. Suppose $G = \bar{A} * \bar{B}$. Then every element x of G can be written uniquely in the form $x = x_A + x_B$ with $x_A \in \bar{A}$ and $x_B \in \bar{B}$. Define $T:G \to G$ by $x \to x_A - x_B$. Then T is clearly an involutory, isometric group automorphism. Let $A = \{T(x)^+ \mid x \in G^+\}$, $B = \{T(x)^- \mid x \in G^+\}$, $[A] = A - A$, and $[B] = B - B$. If $x_A \in \bar{A}^+$, $x_B \in \bar{B}^+$ then $x_A = T(x_A) = T(x)^+ \in A$ and $x_B = -T(x_B) = T(x_B)^- \in B$. Thus, $\bar{A} = [A]$ and $\bar{B} = [B]$.

EXAMPLE 1. Let $C[0,1]$ be the set of continuous functions on the closed interval $[0,1]$. Then $C[0,1]$ forms an abelian lattice ordered group under pointwise addition and order. It is clear that $C[0,1]$ has no non-trivial cardinal summands and so the only involutory, isometric group automorphisms are $T_1(x) = x$ and $T_2(x) = -x$ where $x \in C[0,1]$. Thus, all isometries are of the form $f_1(x) = x + a$ or $f_2(x) = -x + a$ for all $x \in C[0,1]$ and fixed $a \in C[0,1]$.

EXAMPLE 2. It is known that a free abelian lattice ordered group has nontrivial cardinal summands if and only if it is of rank one (1). Thus, if G is such a group of rank greater than one, then its isometries are all of the form $f_1(x) = x + a$ or $f_2(x) = -x + a$ as in Example 1.

EXTENDING CONGRUENCES TO ISOMETRIES

In considering congruences defined on sublattices of G, it is natural to ask when these congruences can be extended to all of G. Proposition 5 characterizes those

congruences which can be extended to isometries while Theorem 6 gives necessary and sufficient conditions on the abelian lattice ordered group for all congruences to be extendable to isometries.

Let L be a sublattice of G containing 0. For a congruence $f:L \to G$ let $T_f:L \to G$ be defined by $T_f(x) = f(x) - f(0)$. Let $A = \{T_f(x)^+ \mid x \in L^+\}$ and $B = \{T_f(x)^- \mid x \in L^+\}$. Using this notation we get the following results.

PROPOSITION 5. A congruence $f:L \to G$ can be extended to an isometry $\bar{f}:G \to G$ if and only if $G = \bar{A} * \bar{B}$ where $\bar{A} \supseteq A$ and $\bar{B} \supseteq B$.

PROOF. (i) Suppose f can be extended to an isometry $\bar{f}:G \to G$. Let $\bar{T}:G \to G$ be defined by $\bar{T}(x) = \bar{f}(x) - \bar{f}(0)$. Then $\bar{T}$ agrees with T_f on L. Let $A_1 = \{\bar{T}(x)^+ \mid x \in G^+\}$, $B_1 = \{\bar{T}(x)^- \mid x \in G^+\}$, $\bar{A} = A_1 - A_1$, and $\bar{B} = B_1 - B_1$. Then $\bar{A} \supseteq A$, $\bar{B} \supseteq B$, $\bar{A} \cap \bar{B} = \{0\}$, and both $\bar{A}$ and $\bar{B}$ are convex l-subgroups. As in the proof of Theorem 3 we get $\bar{T}(x_\alpha) = x_\alpha$ or $\bar{T}(x)_\alpha = -x_\alpha$ for all $x \in G$, $\alpha \in A$, so $x = \bar{T}(x)^+ + \bar{T}(x)^-$ for all $x \in G^+$. Thus, $G = \bar{A} * \bar{B}$.

(ii) Suppose $G = \bar{A} * \bar{B}$ where $\bar{A} \supseteq A$ and $\bar{B} \supseteq B$. Every $x \in G^+$ can be written uniquely as $x = x_A + x_B$ where $x_A \in \bar{A}$ and $x_B \in \bar{B}$. Define $\bar{f}(x_A) = f(0)_A + x_A$, $\bar{f}(x_B) = f(0)_B - x_B$, and $\bar{f}(x) = \bar{f}(x_A) + \bar{f}(x_B)$. Then for $x, y \in G$, $|x-y| = |x_A + x_B - y_A - y_B| = |x-y|_A + |x-y|_B = |x-y|_A + |y-x|_B = |(x-y)_A + (y-x)_B| = |x_A - y_A + y_B - x_B| = |(f(0) + x_A - x_B) - (f(0) + y_A - y_B)| = |\bar{f}(x) - \bar{f}(y)|$. Thus, $\bar{f}:G \to G$ is an isometry. If $x \in L$ then $x_A \in A$ and $x_B \in B$ so $f(x) = T_f(x) + f(0) = T_f(x_A) + T_f(x_B) + f(0)_A + f(0)_B = x_A - x_B + f(0)_A + f(0)_B = \bar{f}(x)$. Hence $\bar{f}$ extends f.

The above proposition can be applied to characterize those lattice ordered groups with the property that every congruence can be extended to an isometry.

THEOREM 6. Let G be an abelian lattice ordered group. Then the following are equivalent.

(1) Every congruence $f:L \to G$, where L

is a sublattice of G containing 0, can be extended to an isometry.

(2) G is strongly projectable.

PROOF. (1) $\Rightarrow$ (2). If $S \subseteq G$ then $S^\perp * S^{\perp\perp} \subseteq G$. For $x \in (S^\perp)^+$, $y \in (S^{\perp\perp})^+$ let $f(x+y) = x - y$. Then $f:S^\perp * S^{\perp\perp} \to G$ is a congruence and so it can be extended to an isometry $\bar{f}:G \to G$. But then $G = \bar{A} * \bar{B}$ where $\bar{A} \supseteq \{f(x)^+ \mid x \in (S^\perp * S^{\perp\perp})^+\} = (S^\perp)^+$ and $\bar{B} \supseteq \{f(x)^- \mid x \in (S^\perp * S^{\perp\perp})^+\} = (S^{\perp\perp})^+$ by Proposition 5. Therefore, $\bar{B} = \bar{A}^\perp \subseteq (S^\perp)^\perp = S^{\perp\perp}$ and $\bar{A} = \bar{B}^\perp \subseteq (S^{\perp\perp})^\perp = S^\perp$ so $G = S^\perp * S^{\perp\perp}$.

(2) $\Rightarrow$ (1). Suppose G is strongly projectable and $f:L \to G$ is a congruence. Then $G = A^\perp * A^{\perp\perp}$ where $A = \{T_f(x)^+ \mid x \in L^+\}$. Since $A^\perp \supseteq B = \{T_f(x)^- \mid x \in L^+\}$ and $A^{\perp\perp} \supseteq A$, Proposition 5 says that f can be extended to an isometry of G.

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