MIXED RECTANGULAR FINITE ELEMENTS FOR PLATE BENDING

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The paper describes three different rectangular plate bending elements based on Reissner's type stationary variational principles. They differ in the number of dependent variables approximated independently and also in the number of nodes per element. The first element types treats the transverse deflection and the three moments as unknowns at each of the corner nodes; the second element type treats the transverse deflection and two normal moments as unknowns at the corner nodes; the third one treats the transverse deflection as unknown at the corner node, and the moments m_x and m_y at the midnodes of opposite sides of the rectangle. These three types of elements are used to solve square plate problems with various boundary conditions and loadings.

INTRODUCTION

Owing to the severe continuity requirements placed on the trial functions employed in the conventional (conforming) finite element models of plate bending (derived with the total potential energy principle), the resulting element matrices are algebraically complex and consequently their solutions require large amounts of computing time. Further, the moments (or stresses) computed using the conventional plate bending elements are not accurate. To avoid these problems Herrmann (1) suggested a new triangular plate bending element which treats the transverse deflection and normal and tangential moments as unknown dependent variables. This element was derived using Reissner's variational principle for a thin plate bending element. Reissner's variational principle yields, as Euler equations, the moment-equilibrium equations and the kinematic relations connecting the transverse displacement (and its derivatives) to the moments. These equations are lower order (2nd order) compared to the fourth-order (biharmonic) equation governing the transverse displacement of the plate. This attractive feature relaxes the continuity requirements on the trial functions. Several mixed finite element models have been derived based on variants of Reissner's functional and/or using various order polynomial approximations (see, for example, (2-6)). The present paper describes construction and applications of three mixed rectangular finite elements.

FORMULATION

Let the middle plane of the plate to be analyzed be denoted $\Omega \subset R^2$ with piecewise smooth boundary $\partial\Omega$. This region is decomposed into finite elements $\{\Omega_e\}_{e}^{N} = 1$. Let w denote the transverse deflection, m_x, m_y, and m_{xy} the moments, and P the lateral load in the plate. The kinematic relations are given by

$$\frac{\partial^2 w}{\partial x^2} = -S(m_x - \nu m_y)$$
$$\frac{\partial^2 w}{\partial y^2} = -S(m_y - \nu m_x) \qquad \text{Eq. 1}$$
$$2\frac{\partial^2 w}{\partial x \partial y} = -2(1+\nu) S\, m_{xy}$$

where $S = 12/Et^3$, E being the Young's modulus, t the thickness, and μ is the Poisson's ratio of the plate. The equilibrium equation is given by

$$-\left(\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2}\right) = P \qquad \text{Eq. 2}$$

These equations must be adjoined by appropriate conditions on the boundary of the plate. We introduce the notation

$$m_n = m_x n_x^2 + 2m_{xy} n_x n_y + m_y n_y^2;$$
$$m_{ns} = -(m_x - m_y) n_x n_y + m_{xy}(n_x^2 - n_y^2); \quad Q_n = q_n + \frac{\partial m_{ns}}{\partial s} \qquad \text{Eq. 3}$$
$$q_n = \left(\frac{\partial m_n}{\partial x} + \frac{\partial m_{xy}}{\partial y}\right) n_x + \left(\frac{\partial m_{xy}}{\partial x} + \frac{\partial m_y}{\partial y}\right) n_y$$
$$\frac{\partial}{\partial n} \equiv n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x} \equiv n_x \frac{\partial}{\partial y} - n_y \frac{\partial}{\partial x}$$

where $n_x = \cos(n,x)$ and $n_y = \cos(n,y)$ are the direction cosines of the outward normal $n = (n_x, n_y)$ on $\partial\Omega$. We specify the following set of boundary conditions.

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(a) essential boundary conditions:

$$w = \hat{w} \text{ on } \partial\Omega_w \text{ , } m_n = \hat{m}_n \text{ on } \partial\Omega_m \qquad \text{Eq. 4}$$

(b) natural boundary conditions:

$$Q_n = \hat{Q}_n \text{ on } \partial\Omega - \partial\Omega_w \text{ , } \frac{\partial w}{\partial n} = \hat{w}_n \text{ on } \partial\Omega - \partial\Omega_m \qquad \text{Eq. 5}$$

Here variables with "^" denote specified values, and $\partial\Omega_w$ and $\partial\Omega_m$ are disjoint sets whose union is $\partial\Omega$.

A variational formulation of Equations 1-5 has been derived (7) and is given by

$$R_1(w, m_x m_y, m_{xy}) = \sum_{e=1}^{N} R_1^e(w^e, m_x^e, m_y^e, m_{xy}^e)$$

Here R_1^e is the restriction of the functional R_1 to element e, and

$$R_1^e(w, m_x, m_y, m_{xy}) = \iint_{\Omega_e} \left\{ -\frac{S}{2}\left[m_x^2 + m_y^2 - 2\nu m_x m_y + 2(1+\nu)m_{xy}^2\right] + \frac{\partial w}{\partial x}\left(\frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x}\right) - Pw \right\} dxdy - \int_{\partial\Omega_e} m_{ns}\frac{\partial w}{\partial s}\, dS - \int_{\partial\Omega - \partial\Omega_m} m_n \hat{w}_n\, dS - \int_{\partial\Omega - \partial\Omega_w} \left(\hat{Q}_n w - \hat{m}_{ns}\frac{\partial w}{\partial s}\right) dS \qquad \text{Eq. 6}$$

wherein for the sake of brevity the element label 'e' is omitted. This functional can be used to construct independent approximations of w, m_x, m_y and m_{xy} . If we assume that the third equation in Equation 1 is identically satisfied (i.e., eliminating m_{xy}), we obtain from R_1^e,

$$R_2^e(w, m_x, m_y) = \iint_{\Omega_e} \left\{ -\frac{S}{2}\left[m_x^2 + m_y^2 - 2\nu m_x m_y\right] + \frac{1}{(1+\nu)S}\left(\frac{\partial^2 w}{\partial x \partial y}\right) + \frac{\partial w}{\partial x}\frac{\partial m_x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial m_y}{\partial y} - Pw \right\} dxdy \qquad \text{Eq. 7}$$

wherein the boundary terms are omitted temporarily. Functional in Equation 7 can be used to construct independent approximations of w, m_x and m_y.

Mixed Model I.

Functional R_1 is employed to construct the rectangular finite element. Bilinear approximations are used for each variable. Thus the element has four nodes and four degrees of freedom at each node (see Figure 1a), resulting in a 16 by 16 element stiffness matrix. The output contains w, m_x, m_y, and m_{xy} at each nodal point.

Mixed Model II.

Here functional R_2 is used to develop the element. Again bilinear approximations are employed for each of the three variables. At each of the four corner nodes there exist three (w, m_x, and m_y) degrees of freedom (see Figure 1b) resulting in a 12 by 12 element stiffness matrix. The quadratic element contains eight nodes with three degrees of freedom per node.

Mixed Model III.

This element is based on functional R_2. Here bilinear approximations are used for the transverse displacement w, and linear approximations for m_x and m_y. The four corner nodes each have one displacement degree of freedom. Midnodes on the sides perpendicular to the y-axis each have one degree of freedom m_y, and midnodes on the sides perpendicular to the x-axis each have one degree of freedom m_x per node (see Figure 1c). This element results in a 8 by 8 element stiffness matrix. Note that m_x is linear along x but constant along y and m_y is linear along y and constant along x.

The trial functions (or approximating functions) are shown in Figure 1. For lack of space the element stiffness matrices for each type are not given here. The element matrices are assembled in the usual manner (see Zienkiewicz (8) and Oden and Reddy (9), and the resulting global set of equations are solved for the unknown nodal values.

NUMERICAL RESULTS

The above three types of mixed rectangular elements are now used to solve simple problems. Square plates (of length L) with three types of edge conditions, simply-supported, clamped, and two opposite edges simply-supported and the other two clamped, are solved for uniformly distributed load and concentrated load at the center of the plate. The results are compared with each other and also with those from a hybrid model of Allman (10) and a compatible displacement model of Clough and Tocher (11). Table 1 shows a comparison of center displacements and bending moments for a simply-supported plate with uniform loading. In Table 2 central deflection and bending moments for a clamped square plate under uniform loading are compared with those for the mixed model IV of Herrmann (1), hybrid model of Allman (10), and conventional cubic displacement model of Clough and Tocher (11). Table 3 contains results for the same plate under concentrated load at the middle. In Table 4 results are presented for a simply-supported plate under concentrated load. Finally Table 5 contains values of central deflection and bending moment, and bending moment at the center of side, for a square plate with two opposite sides clamped and the other two simply-supported. Results are given for uniform loading as well as for concentrated load.

An examination of the results presented indicate that the mixed models I, II, and III described herein are giving better ac-

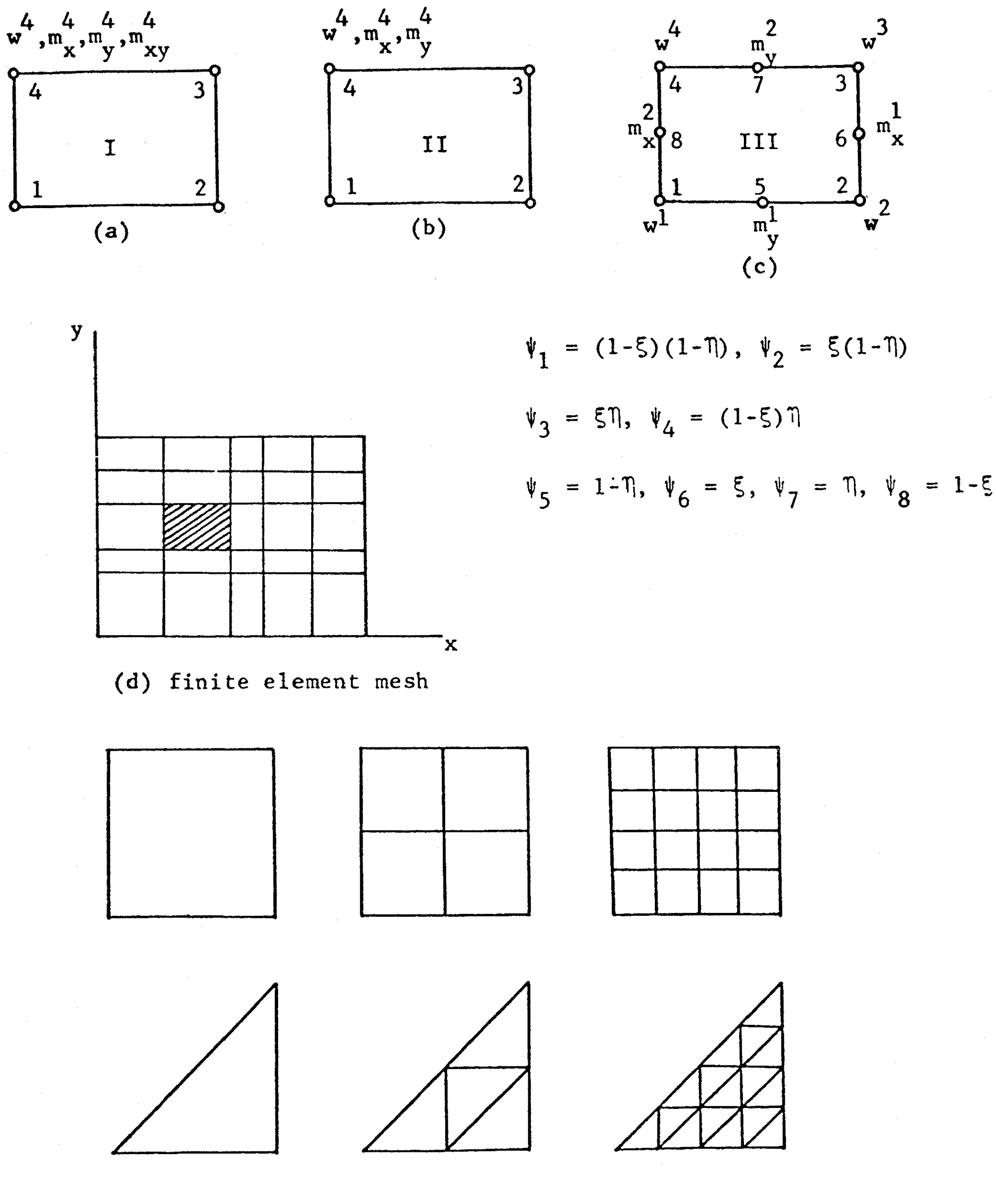

FIGURE 1. Rectangular plate bending elements, finite element mesh of plate, and various mesh types.

TABLE 1. *Simply-supported square plate under uniformly distributed load, P.*

Mesh Size	Mixed Model I	Mixed Model II Linear	Mixed Model II Quadratic	Mixed Model III	Mixed Model IV (1)	Hybrid Model (10)	Compatible Cubic Displ. Model (11)
		Central deflection $WD \times 10^2/PL^4$ (0.4062)[a]					
1×1	0.4613(16)[b]	0.3906(12)	0.3867(24)	0.4943(8)	0.9018(6)	0.347(12)	0.220(12)
2×2	0.4237(36)	0.4082(27)	0.4053(63)	0.4289(21)	0.5127(15)	0.392(27)	0.371(27)
4×4	0.4106(100)	0.4069(75)	0.4062(195)	0.4117(65)	0.4316(45)	0.403(75)	0.392(75)
6×6	0.4082(196)	0.4066(147)	0.4062(399)	0.4087(133)	0.4174(101)	----	----
		Bending moment at the center $M_x \times 10/PL^2$ (0.479)[a]					
1×1	0.7196	0.6094	0.3813	0.3482	0.328	0.604	----
2×2	0.5246	0.5049	0.4818	0.4498	0.446	0.515	----
4×4	0.4891	0.4849	0.4788	0.4721	0.471	0.487	----
6×6	0.4834	0.4815	0.4789	0.4759	0.476	----	----

[a] Exact solution from Reference 13.
[b] Numbers in parenthesis indicate the number of degrees of freedom in the mesh.

TABLE 2. *Clamped square plate under uniformly distributed load, P.*

Mesh Size	Mixed Model I	Mixed Model II Linear	Mixed Model II Quadratic	Mixed Model III	Mixed Model IV (1)	Hybrid Model (10)	Compatible Cubic Displ. Model (11)
		Central deflection $WD \times 10^2/PL^4$ (0.1265)[a]					
1×1	0.1664(16)	0.1563(12)	0.1466(24)	0.2278(8)	0.7440(6)	0.087(12)	0.026(12)
2×2	0.1529(36)	0.1480(27)	0.1260(63)	0.1627(21)	0.2854(15)	0.132(27)	0.120(27)
4×4	0.1339(100)	0.1325(75)	0.1264(195)	0.1359(65)	0.1696(45)	0.129(75)	0.121(75)
6×6	0.1299(196)	0.1292(147)	0.1265(399)	0.1307(133)	0.1463(101)	----	----
		Bending moment at the center $M_x \times 10/PL^2$ (0.231)[a]					
1×1	0.5193	0.4875	0.2056	0.2487	0.208	0.344	----
2×2	0.3166	0.2899	0.2248	0.2432	0.242	0.314	----
4×4	0.2478	0.2443	0.2287	0.2339	0.235	0.250	----
6×6	0.2374	0.2358	0.2290	0.2313	0.232	----	----

[a] Exact solution from Reference 13.

TABLE 3. *Clamped square plate under concentrated load, P_0.*

Mesh Size	Mixed Model I	Mixed Model II Linear	Mixed Model II Quadratic	Mixed Model III	Mixed Model IV (1)	Hybrid Model (10)	Compatible Cubic Displ. Model (11)
		Central deflection $WD \times 10^2/P_0L^2$ (0.561)[a]					
1×1	0.6658(16)	0.6250(12)	0.6416(24)	0.9111(8)	2.2232(6)	0.260(12)	0.176(12)
2×2	0.6927(36)	0.6498(27)	0.5604(63)	0.7340(21)	1.2020(15)	0.515(27)	0.492(27)
4×4	0.6071(100)	0.5925(75)	0.5613(195)	0.6207(65)	0.7759(45)	0.553(75)	0.549(75)
6×6	0.5846(196)	0.5772(147)	0.5613(399)	0.5908(133)	0.6701(101)	----	----
		Bending moment at the corner M_x P_0 (0.1257)					
1×1	0.1600	0.1500	0.1208	0.0995	0.0625	0.1031	----
2×2	0.1075	0.1163	0.13497	0.0951	0.0858	0.1236	----
4×4	0.1227	0.1240	0.1300	0.1155	0.1065	0.1233	----
6×6	0.1241	0.1249	0.11281	0.1208	0.1145	----	----

[a] Exact solution from Reference 13.

TABLE 4. *Simply-supported square plate under concentrated load, P_0.*

Mesh Size	Mixed Model I	Mixed Model II Linear	Mixed Model II Quadratic	Mixed Model III	Mixed Model IV (1)	Hybrid Model (10)	Compatible Cubic Displ. Model (11)
		Central deflection $WD \times 10^2/P_0L^2$ (1.160)[a]					
1×1	1.8450(16)	1.5625(12)	1.122 (24)	1.9770(8)	2.706 (6)	1.042(12)	0.798(12)
2×2	1.3476(36)	1.2813(27)	1.159 (63)	1.3871(21)	1.741 (15)	1.132(27)	1.039(27)
4×4	1.2158(100)	1.1972(75)	1.160 (195)	1.2270(65)	1.351 (45)	1.153(75)	1.130(75)
6×6	1.1873(190)	1.1784(147)	1.160 (399)	1.1927(100)	1.257 (101)	----	----

[a] Exact solution from Reference 13.

TABLE 5. *Clamped simply-supported square plate*

Mesh Size	Under uniformly distributed load, P — Central deflection $WD \times 10^2/PL^4$ (0.192)				Under concentrated load P_0 — Central deflection $WD \times 10^2/P_0L^2$ (0.7071)			
	Mixed Model I	Mixed Model II Linear	Mixed Model II Quadratic	Mixed Model III	Mixed Model I	Mixed Model II Linear	Mixed Model II Quadratic	Mixed Model III
1×1	0.2446	0.2016	0.2034	0.3214	0.9785	0.8065	0.7564	1.2860
2×2	0.2168	0.2059	0.1912	0.2292	0.8468	0.7900	0.7037	0.9041
4×4	0.1987	0.1963	0.1917	0.2009	0.7514	0.7353	0.7042	0.7652
6×6	0.1963	0.1937	0.1917	0.1958	0.7279	0.7200	0.7041	0.7341
	Bending moment at the center $M_X \times 10/PL^2$ (0.332)				Bending moment at the corner M_X/P_0 (0.166)			
1×1	0.6752	0.5565	0.2680	0.3792	0.2348	0.1790	0.1619	0.1291
2×2	0.4247	0.3867	0.3254	0.3623	0.1506	0.1601	0.1735	0.1437
4×4	0.3526	0.3477	0.3321	0.3417	0.1638	0.1648	0.1700	0.1577
6×6	0.3452	0.3394	0.3324	0.3367	0.1648	0.1654	0.1682	0.1619

curacies for the displacement and moments than the mixed model of Herrmann (1), the hybrid model of Allman (10), and the compatible cubic displacement models of Clough and Tocher (11) and Fraeijs de Veubeke and Sander (12). Among the three mixed models described here, mixed model II gives the best accuracies. Although mixed model I gives better accuracies than mixed model III, it requires more storage and computational times. Thus there is a compromise between accuracy and the computational time involved.

SUMMARY AND CONCLUSIONS

Three types of rectangular plate bending finite elements are described and compared with each other and also with other mixed, hybrid and compatible finite element models in terms of accuracy and the number of unknowns used in each mesh (which is proportional to the computational time). Square plates with various edge conditions and loadings are analyzed numerically using all three models. It is concluded from the present numerical analysis that the models described herein are economical and give more accurate results. Another advantage which cannot be judged from the numerical results is the very little amount of time needed to compute the element matrices *exactly*.

Application of these elements to vibration and stability of plates is under way and results will appear elsewhere. Extensions to orthotropic or more general anisotropic plates can be done with very little effort. Use of similar formulation for large deflection analysis of plates is straightforward.

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