

INTERFACIAL SHEAR-STRESS DISTRIBUTION AT A FINITE-LENGTH FIBER SUBJECTED TO AN ARBITRARY PLANE-STRESS FIELD

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Using the elementary shear-lag theory for an isolated fiber embedded in a matrix, the interfacial shear-stress distribution is calculated as a function of the angle between the loading direction and the fiber axis. The results obtained show reasonable agreement with available photoelastic data.

When a composite containing continuous fibers is loaded parallel to the fibers, the matrix and fibers are strained the same amount. However, when a fiber fractures due to presence of flaws, each fiber segment and the matrix generally experience quite different strain distributions. If the longitudinal fiber modulus, E_f , is greater than the matrix elastic modulus, E_m , the fiber segment restrains elongation of the matrix in the local region adjacent to the matrix, as shown in Figure 1. The elastic strain dis-

interfacial shear stress at a distance, x , from the end of the fiber is denoted as $\tau(x)$. The total force accumulated along the fiber up to a distance x_1 from the end is given by

$$P(x) = 2\pi r \int_0^{x_1} \tau(x) dx \quad \text{Eq. 1}$$

The axial normal stress in the fiber segment, $\sigma(x)$, is equal to the axial force divided by the fiber cross-sectional area, i.e.,

$$\sigma(x) = P(x)/\pi r^2 = (2/r) \int_0^{x_1} \tau(x) dx \quad \text{Eq. 2}$$

Numerous analytical and experimental investigations have been devoted to study of the nature and mechanisms of fiber-matrix interaction and load transfer, cf. the simple analyses presented by Cox (1), Dow (2), and Rosen (3).

The analysis presented here is a generalization of the work of Cox (1) from loading parallel to the fiber (σ_1) to arbitrary in-plane loading ($\sigma_1, \sigma_2, \sigma_3$).

ANALYSIS

First, the matrix is considered to act alone under the influence of a general in-plane stress field. The resulting displacement component in the direction of the fibers is denoted by u_m^0 . Next the fibers are considered to be present and their axial displacement is denoted by u . The force, P , developed in the fiber is then related to the difference between the two displacements as follows:

$$P_{,x} = H(u - u_m^0) \quad \text{Eq. 3}$$

where H is a constant, x is the axial position along the fiber, and $P_{,x}$ denotes $\delta P/\delta x$.

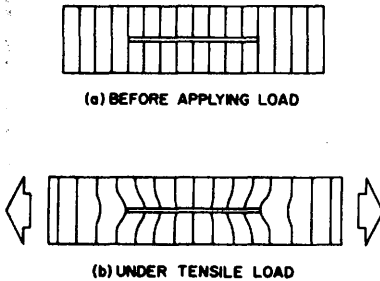


FIGURE 1. Schematic diagram depicting the deformation occurring in an elastic matrix adjacent to a finite-length fiber.

tribution in the matrix is non-uniform, with the non-uniformity decreasing with increasing distance from the fiber. This same phenomenon occurs in composites containing discontinuous fibers, such as whiskers.

It has been shown (1) that the normal-stress distribution in a discontinuous fiber is determined by the distribution of shear stress at the fiber-matrix interface. A fiber segment of radius r is considered, and the

However, $P = A_f (E_f - E_m) u_{,x}$ Eq. 4

where $A_f \equiv$ fiber cross-sectional area, $E_f \equiv$ fiber longitudinal elastic modulus, and $E_m \equiv$ matrix elastic modulus.

Equations 3 and 4 can be combined with the following result:

$$P_{,XX} - \beta^2 P = -H e_m^0 \quad \text{Eq. 5}$$

where $e_m^0 = u_{m,x}^0 = \text{constant}$ and

$$\beta^2 \equiv (H/A_f) (E_f - E_m)^{-1}$$

The general solution of Equation 5 is

$$P(x) = C_1 \cosh \beta x + C_2 \sinh \beta x + A_f (E_f - E_m) e_m^0 \quad \text{Eq. 6}$$

The boundary conditions which must be satisfied for P are that it must be equal to zero at each end of the fiber, i.e., $P(0) = P(L) = 0$, where $L \equiv$ length of fiber segment. Then from Equations 2 and 6, the axial stress in the fiber is

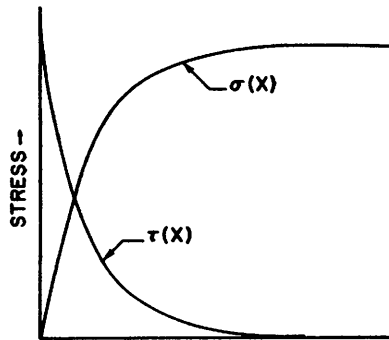
$$\sigma(x) = P/A_f = (E_f - E_m) e_m^0 \left[1 - \frac{\cosh \beta (L-x)}{\cosh (\beta L/2)} \right] \quad \text{Eq. 7}$$

and the interfacial shear stress is, from Equation 2,

$$\tau(x) = (\tau/2) \sigma_{,x} = (E_f - E_m) (\beta e_m^0 \tau/2) \frac{\sinh \beta (L-x)}{\cosh (\beta L/2)} \quad \text{Eq. 8}$$

Figure 2 shows plots of the stress distributions as given by Equations 7 and 8.

The analyses of Dow (2) and Rosen (3) differ from that of Cox (1) only in the expressions for the coefficients. Table 1 lists the various expressions in dimension-



DISTANCE FROM END OF FIBER, X--

FIGURE 2. Stress distributions along the length of a discontinuous fiber.

less form. Cox assumed that the discontinuous fiber was surrounded by six hexagonally packed adjacent fibers. Rosen used a self-consistent model consisting of a fiber surrounded by a sheath of matrix material, which in turn is embedded in an infinite medium having elastic properties equal to that of the composite.

The above mentioned analyses have been confirmed qualitatively by photoelastic studies (cf. Ref. 4) of matrix material in the vicinity of a single, discontinuous fiber when the composite is loaded in the direction of the fiber.

TABLE 1. Expressions for the dimensionless parameter βd according to three different elastic load-transfer theories.^a

Analysis	Ref.	βd
Cox	1	$[8G_m / (E_f - E_m)]^{1/2} [1n(\sqrt{3} V_f / 2\pi)]^{1/2}$
Dow	2	$2 \{ 2\sqrt{2} (G_f / E_f) [1 + (E_f V_f / E_m V_m)] \}^{1/2} \cdot \{ (\sqrt{2}-1) + (G_f / G_m) [\sqrt{2 + (V_m / V_f)} - \sqrt{2}] \}^{-1/2}$
Rosen	3	$2(G_m / E_f)^{1/2} [\sqrt{V_f} / (1 - \sqrt{V_f})]^{1/2}$

^a Here G_f and G_m are the respective fiber and matrix shear moduli, and V_f and V_m are the respective fiber and matrix volume fractions.

Apparently the only experiments on the stress distribution for uniaxial loading at various acute angles to the fiber direction are those of McGarry and Fujiwara (5). For this case, the strain parallel to the fiber (ϵ_m^0) is related to the applied stress (σ_1') at angle θ from the fiber direction as follows:

$$\epsilon_m^0 = \epsilon_1 = (\sigma_1' / E_{11}) (\cos^2 \theta - \nu_{12} \sin^2 \theta) \quad \text{Eq. 9}$$

where ν_{12} and E_{11} are the major Poisson's ratio and major Young's modulus of the composite. Thus, combining Equations 8 and 9, one obtains the following expression for the interfacial shear stress $\tau(\theta)$, at a given location x , as a function of the load orientation angle θ :

$$\tau(\theta) = \tau(0) (\cos^2 \theta - \nu_{12} \sin^2 \theta) \quad \text{Eq. 10}$$

Figure 3 shows a comparison between McGarry and Fujiwara's photoelastic results and the prediction by use of Equation 10, assuming that the composite has a major Poisson's ratio of 0.26. Although the agreement is only fair, it is a considerable im-

provement over the $\cos^2 \theta$ relation suggested by McGarry and Fujiwara.

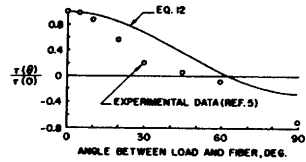


FIGURE 3. Shear stress distribution versus angular orientation of uniaxial load.

REFERENCES

1. H. L. COX, *Brit. J. Appl. Phys.* 3: 72-79 (1952).
2. N. F. DOW, *Study of Stresses Near a Discontinuity in a Filament-Reinforced Composite Metal*, General Electric Co., Space Sciences Lab., Philadelphia, Pa., Rept. R63SD61, Aug., 1963.
3. B. W. ROSEN, *Fiber Composite Materials*, Amer. Soc. For Metals, Metals Park, O., 1965, pp. 38-58.
4. D. M. SCHUSTER and E. SCALA, *Trans. AIME* 230: 1635-1640 (1964).
5. F. J. MCGARRY and M. FUJIWARA, *Modern Plastics* 45 (11): 143 ff. (July, 1968).