FLEXURAL MICROMECHANICS OF A COMPOSITE MATERIAL CONTAINING LARGE-DIAMETER FIBERS

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Using the elementary theory of composite beams and Reissner's variational principle, expressions are derived for the flexural stiffness and the thickness-shear flexibility of a composite material containing a single row of uniformly spaced longitudinal filaments with circular cross sections. The only input data used are the geometry and elastic moduli of the constituent materials. These single-layer flexural stiffness and thickness-shear flexibility values may be used to predict the behavior of a beam arbitrarily laminated. Numerical results are presented for boron/epoxy, S-glass/epoxy, and boron-aluminum composites.

The macroscopic properties of a single layer of filamentary composite material can be determined experimentally. However, due to the large number of tests required, it is desirable to be able to calculate these properties from knowledge of the geometry and the constituent-material properties. This has been the impetus for numerous micromechanics analyses, such as those summarized in (1).

In beams, an important physical quantity is the flexural rigidity:

$$D = \int_A z^2 E(z)\, dA \qquad \text{Eq. 1}$$

where A = cross-sectional area, z = distance from midplane, E(z) = longitudinal Young's modulus. In laminated rectangular-section beams, it has been customary to assume that each layer is homogeneous through its thickness, so that the integral appearing in Equation 1 can be replaced by a summation as follows (2) :

$$D = (\bar{W}/3) \sum_{k=1}^{n} E_k \left(z_k^3 - z_{k-1}^3\right) \qquad \text{Eq. 2}$$

where k refers to the kth ply, n = number of plies, $\bar{W}$ = beam width; z_k and z_{k-1} are the values of z associated with the upper and lower surfaces of the kth ply.

For composites containing many very small-diameter fibers distributed more or less randomly through the thickness, Equation 2 would be expected to be valid. This has been borne out for glass-fiber/epoxy (0.0004 in. diameter fibers) by Tsai (3). However, so-called monofilament composites, which contain only one row of fibers per ply, are coming into use. Large-diameter fibers of boron (4) are used most extensively, but S-glass has also been used (5). For laminated beams consisting of only a few of these single-filament-row layers, Equation 2 would not be expected to hold because of the large amount of low-modulus matrix material located at appreciable relative distances from the midplane of each layer. This is demonstrated quantitatively in the micromechanics analysis presented here. The only previous works along this line are the very approximate analyses due to Norris (6) and Margolin (7).

For a long time, it has been suspected that thickness-shear* flexibility is significant in filamentary composites (8, 9). Although some laminate bending analyses have incorporated thickness-shear flexibility (10-14), none of these have presented rational micromechanics bases for determining the required single-layer flexibility. A micromechanics analysis based on the Jourawski shear theory, see (15), is presented here.

FLEXURAL RIGIDITY OF A LAYER

The typical repeating one-quarter cross section shown in Figure 1 is considered. Using the Bernoulli-Euler hypothesis as a first approximation, the bending stress, σ, is calculated as follows:

$$\sigma = \begin{cases} \alpha,_x E_f z & \text{in } a_f \\ \alpha,_x E_m z & \text{in } a_m \end{cases} \qquad \text{Eq. 3}$$

* The terms transverse shear, horizontal shear, and shear due to bending are used by some investigators instead of thickness shear.

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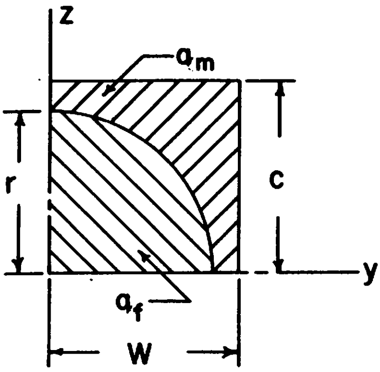

FIGURE 1. Typical repeating one-quarter cross section.

where $\alpha,_x$ = bending curvature, E = Young's modulus, a_f and a_m are cross-sectional areas (Figure 1), and subscripts f and m denote fiber and matrix, respectively.

The bending strain energy in an elemental volume one unit long and having cross-sectional area $a = a_f + a_m$ is:

$$U_b = (1/2) \int_{a_f} (\sigma_f^2/E_f)\, da + (1/2) \int_{a_m} (\sigma_m^2/E_m)\, da \qquad \text{Eq. 4}$$

Thus, one obtains

$$2U_b/\alpha^2,_x = (E_f - E_m) \int_0^r z^2 (r^2 - z^2)^{1/2}\, dz + E_m \int_0^c W z^2\, dz$$

which integrates to yield the following result:

$$U_b/\alpha^2,_x = (E_f - E_m)(\pi r^4/32) + E_m W c^3/6 \qquad \text{Eq. 5}$$

To use Reissner's variational principle (16), the following functional is introduced:

$$\phi_b = \int_{t_1}^{t_2} \left(\int_a \alpha,_x z\sigma\, da - U_b \right) dt \qquad \text{Eq. 6}$$

Reissner's principle states that

$$\delta\, \phi_b = 0 \qquad \text{Eq. 7}$$

The bending moment is defined by

$$m = 4 \int_a z\sigma\, da \qquad \text{Eq. 8}$$

Since the bending curvature is uniform throughout a given cross section, Equations 5-8 can be combined to yield:

$$\delta \int_{t_1}^{t_2} \left\{ (m\alpha,_x/4) - \left[(E_f - E_m)(\pi r^4/32) + (E_m W c^3/6) \right] \alpha^2,_x \right\} dt = 0$$

Thus,

$$m = \left[(E_f - E_m)(\pi r^4/4) + (4E_m W c^3/3) \right] \alpha,_x \qquad \text{Eq. 9}$$

The layer flexural rigidity, $\bar{d}$, is defined as follows:

$$\bar{d} = m/\alpha,_x \qquad \text{Eq. 10}$$

Thus, from Equation 9,

$$\bar{d} = (E_f - E_m)(\pi r^4/4) + (4E_m W c^3/3)$$

or

$$\bar{d} = \left[(3\pi/16)(E_f - E_m)(d/h)^4 (h/\bar{W}) + E_m \right] (\bar{W}h^3/12) \qquad \text{Eq. 11}$$

To show the reduction in flexural rigidity as compared to that implied by Equation 2, it is desirable to evaluate the stretching stiffness, $\bar{a}$, defined as the axial force per unit axial strain. Assuming uniform strain throughout the cross section, one obtains this expression:

$$\bar{a} = \left[(\pi/4)(E_f - E_m)(d/h)^2 (h/\bar{W}) + E_m \right] (\bar{W}h) \qquad \text{Eq. 12}$$

This same result is predicted by the so-called "law of mixtures."

Now a flexural rigidity efficiency factor is defined as follows:

$$\eta_b = 12\, \bar{d}/\bar{a}h^2 \qquad \text{Eq. 13}$$

For a homogeneous material, = 1, as implied by Equation 2.

Substituting Equations 11 and 12 into Equation 13, one obtains:

$$\eta_b = \frac{1 + (3\pi/16)\left[(E_f/E_m) - 1 \right] (d/h)^4 (h/\bar{W})}{1 + (\pi/4) \left[(E_f/E_m) - 1 \right] (d/h)^2 (h/\bar{W})} \qquad \text{Eq. 14}$$

Table 1 gives values of η_D for practical values of d/h and $h/\bar{W}$ and mechanical properties typical of boron/epoxy, S-glass/epoxy, and boron/aluminum. For the ranges of values covered, it can be seen that the effect of d/h is much stronger than the effects of $h/\bar{W}$ and E_f/E_m.

Table 1 also presents values of η_D calculated by the following approximate expression, which is derived by neglecting the contribution of the matrix material to both $\bar{I}$ and $\bar{d}$ (7):

$$\eta_D \approx (3/4)(d/h)^2 \qquad \text{Eq. 15}$$

It is seen that Equation 15 provides a lower bound which increases as E_f/E_m is increased.

THICKNESS-SHEAR FLEXIBILITY OF A LAYER

This analysis uses the Jourawski shear theory, which is presented for the homogeneous case in elementary texts on strength of materials. From Equations 3 and 10, the bending stresses acting on the left and right sides of an element of length Δx (Figure 2) are:

$$\sigma_i^{(L)} = (m/\bar{d})E_i z \ , \ \sigma_i^{(R)} = (m+\Delta m)(E_i/\bar{d})z \qquad \text{Eq. 16}$$

where $i = f$ in a_f and $i = m$ in a_m.

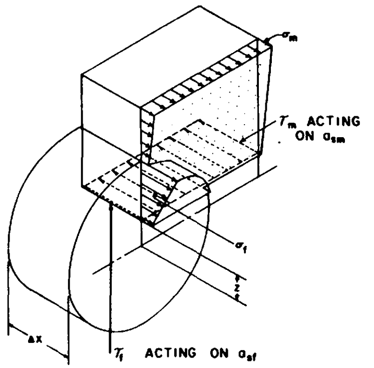

FIGURE 2. Schematic diagram showing equilibrium of bending stresses (solid arrows) and horizontal shear stresses (dotted half-head arrows).

TABLE 1. *Flexural rigidity efficiency of various single-filament-row composites.*

$h/\bar{W}$	d/h	V_f	η_D, Eq. 14: Boron/Epoxy[a]	S-Glass/Epoxy[b]	Boron/Al[c]	Approx. η_D, Eq. 15
0.80	0.80	0.402	0.491	0.529	0.653	0.480
	0.85	0.454	0.549	0.580	0.684	0.542
	0.90	0.508	0.615	0.640	0.718	0.608
	0.95	0.566	0.682	0.701	0.762	0.677
0.85	0.80	0.427	0.490	0.527	0.646	0.480
	0.85	0.482	0.549	0.578	0.678	0.542
	0.90	0.540	0.614	0.638	0.713	0.608
	0.95	0.602	0.681	0.699	0.755	0.677
0.90	0.80	0.452	0.490	0.525	0.640	0.480
	0.85	0.511	0.549	0.577	0.670	0.542
	0.90	0.572	0.614	0.636	0.708	0.608
	0.95	0.637	0.681	0.699	0.752	0.677
0.95	0.95	0.673	0.681	0.697	0.751	0.677
1.00	1.00	0.785	0.753	0.764	0.800	0.750
1.05	0.95	0.744	0.681	0.696	0.747	0.677

[a] $E_f/E_m = 120$; [b] $E_f/E_m = 24$; [c] $E_f/E_m = 6$.

Since these bending stresses act on identical cross-sectional areas, the horizontal shear force acting on the bottom of the element is:

$$F_h = \int_{a_o} \left[\sigma_i^{(R)} - \sigma_i^{(L)} \right] da \qquad \text{Eq. 17}$$

where area of integration, a_o, is shown cross-hatched in Figure 3.

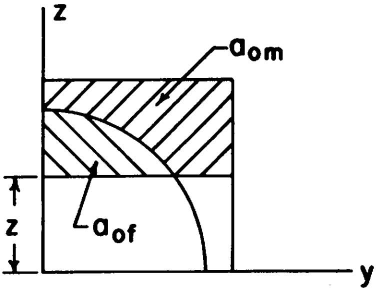

FIGURE 3. Schematic diagram showing thickness shear area. $a_o = a_{of} + a_{om}$.

Combining Equations 16 and 17, one obtains the following:

$$F_h = (\Delta m/\bar{I})\, Y \qquad \text{Eq. 18}$$

where

$$Y = \int_{a_o} E_i z\, da = \int_z^c (E_f b_f + E_m b_m)\, z\, dz \qquad \text{Eq. 19}$$

The varying widths, b_f and b_m, are given by the following expressions:

$$b_f = (r^2 - z^2)^{1/2}, \quad b_m = W - (r^2 - z^2)^{1/2} \quad ; \; 0 \le z \le r$$
$$b_f = 0, \quad b_m = W \; ; \qquad r \le z \le c \qquad \text{Eq. 20}$$

Using Equations 20 in Equation 19 and evaluating the integrals, one obtains:

Eq. 21

$$Y = (1/3)(E_f - E_m)(r^2 - z^2)^{3/2} + (E_m W/2)(c^2 - z^2); \quad 0 \le z \le r$$
$$Y = (E_m W/2)(c^2 - z^2) \; ; \qquad r \le z \le c$$

The horizontal force F_h must equilibrate the horizontal shear stresses, τ_f and τ_m, acting on the bottom face of the element (see Figure 2):

$$F_h = \tau_f a_{sf} + \tau_m a_{sm} \qquad \text{Eq. 22}$$

where the shear areas are given by:

$$a_{sf} = b_f \Delta x, \quad a_{sm} = b_m \Delta x \qquad \text{Eq. 23}$$

For the two constituent materials, Hooke's law in shear can be expressed as follows:

$$\tau_f = G_f \gamma, \quad \tau_m = G_m \gamma \qquad \text{Eq. 24}$$

where γ = engineering shear strain. It is noted that the thickness-shear stresses must be equal to the horizontal shear stresses in order to maintain rotatory equilibrium.

Combining Equations 18 and 22-24, one obtains the following general expression for the thickness-shear-strain distribution throughout the cross section:

$$\gamma = (G_f b_f + G_m b_m)^{-1} (Y/\bar{I})(\Delta m/\Delta x) \qquad \text{Eq. 25}$$

The layer thickness shear force, q, is defined as follows:

$$q = 4 \int_{a_f} \tau_f\, da + 4 \int_{a_m} \tau_m\, da \qquad \text{Eq. 26}$$

Substituting Equations 21, 24, and 25 into Equation 26 and performing rather laborious integrations, one obtains

$$q = \Delta m/\Delta x \qquad \text{Eq. 27}$$

which, of course, is necessary in static beam theory. This serves as a check on the analysis up to this point.

The thickness-shear strain energy in an elemental one-quarter-cross-section volume one unit long is

$$U_s = (1/2) \int_{a_f} G_f \gamma^2\, da + (1/2) \int_{a_m} G_m \gamma^2\, da \qquad \text{Eq. 28}$$

Substituting Equations 20, 21, and 25 into Equation 28 yields the following result:

$$2U_s(\bar{I}/q)^2 = (G_f - G_m) \int_0^r \left[\frac{(1/3)(E_f - E_m)(r^2 - z^2)^{3/2} + (E_m W/2)(c^2 - z^2)}{(G_f - G_m)(r^2 - z^2)^{1/2} + G_m W} \right]^2 \cdot (r^2 - z^2)^{1/2}\, dz$$
$$+ G_m \int_0^r \left[\frac{(1/3)(E_f - E_m)(r^2 - z^2)^{3/2} + (E_m W/2)(c^2 - z^2)}{(G_f - G_m)(r^2 - z^2)^{1/2} + G_m W} \right]^2 W\, dz \qquad \text{Eq. 29}$$
$$+ G_m \int_r^c (E_m/2G_m)^2 (c^2 - z^2)^2\, W\, dz$$

Although the first two integrals on the right-hand side of Equation 29 can be integrated numerically for specific values of the geometric and material parameters of interest, it does not appear possible to evaluate them in closed form. Since $(G_f - G_m) >> G_m$ for composites of technical interest, one might consider omitting the term $G_m W$ in the denominators of the two integrals under discussion. Unfortunately, however, when this simplification is made, the second integral increases without bound except in the case when $r = c$, which is not desirable in practice due to fiber contact problems.

To facilitate the numerical evaluation of the shear flexibility, it is convenient to introduce the following dimensionless quantities:

$$\rho = r/c = d/h, \quad \zeta = z/c$$
$$F = 2G_m U_s \, (\bar{d}/q\, E_m c^3)^2 \qquad \text{Eq. 30}$$

Then Equation 29 becomes

$$F = F_1 + F_2 \qquad \text{Eq. 31}$$

where

$$F_1 = \int_0^\rho \frac{[\,(1/3)(E_f E_m^{-1} - 1)(\rho^2 - \zeta^2)^{3/2} + (1/2)(\bar{W}/h)(1-\zeta^2)\,]^2}{(G_f G_m^{-1} - 1)\,(\rho^2 - \zeta^2)^{1/2} + (\bar{W}/h)}\, d\zeta$$

$$F_2 = \int_\rho^1 (1/4)\,(\bar{W}/h)\,(1-\zeta^2)^2\, d\zeta$$

Performing the integration to obtain F_2 yields the following closed-form expression:

$$F_2 = (1/4)(\bar{W}/h)\,[(8/15) - \rho + (2/3)\rho^3 - (1/5)\,\rho^5\,] \qquad \text{Eq. 32}$$

The Reissner functional for this problem is:

$$\phi_s = \int_{t_1}^{t_2} \left[\int_{a_f} (w,_x + \alpha)\, \tau_f \, da + \int_{a_m} (w,_x + \alpha)\, \tau_m \, da - U_s \right] dt$$

Setting the variation of ϕ_s equal to zero and using the definition of F in Equation 30, one obtains:

$$\delta \int_{t_1}^{t_2} [\,(w,_x + \alpha)(q/4) - (F/2G_m)(qE_m c^3/\bar{d})^2\,]\, dt = 0$$

Thus, $(1/4)\,(w,_x + \alpha) - (qF/G_m)(E_m c^3/\bar{d})^2 = 0$

or $$q = (\bar{d}/E_m c^3)^2\,(1/4F)\,(w,_x + \alpha)$$

Eq. 33

The thickness-shear flexibility, s, is defined as follows:

$$s = (w,_x + \alpha)/q \qquad \text{Eq. 34}$$

Thus, from Equations 33 and 34, one obtains

$$s = 4F\,(E_m c^3/\bar{d})^2 \qquad \text{Eq. 35}$$

For a homogeneous rectangular-section beam made of the same material used as the matrix in the composite, application of Reissner's principle gives

$$s_h = (6/5)\,/\,G_m \bar{W}\, h \qquad \text{Eq. 36}$$

A thickness-shear flexibility factor, η_s, is defined as follows:

$$\eta_s = s_h/s \qquad \text{Eq. 37}$$

The composite flexibility, s, is placed in the denominator of Equation 37 because a small value of s results in the most desirable composite, *i.e.* a stiff one.

It is convenient to introduce the following dimensionless factor:

$$\beta = \bar{a}/E_m \bar{W} h = 1 + (\pi/4)\,(E_f E_m^{-1} - 1)\,(d/h)^2 (h/\bar{W}) \qquad \text{Eq. 38}$$

Combining Equations 14 and 35-38, one obtains the following result:

$$\eta_s = (2/15)\,(\bar{W}/h)\,(\beta^2/F) \qquad \text{Eq. 39}$$

Using the typical constituent-material properties and geometrical parameters listed in Table 2, numerical calculations of η_s were carried out for boron/epoxy, S-glass/epoxy, and boron/aluminum. Results are shown in Table 3. It is noted that η_s varies quite widely among the three typical composite materials considered.

CONCLUSION

Using strength-of-materials theory, a micromechanics analysis is presented for a single-filament-row beam. In conjunction with Reissner's principle, the results of the micro stress analysis are used to derive equations for the flexural rigidity and thickness-shear flexibility. Numerical results are presented for boron/epoxy, S-glass /epoxy, and boron/aluminum. At the expense of greater computational complexity, the analysis can be extended to include other fiber cross-sectional shapes such as hollow ones, anisotropic filament material, and statistical variations, such as nonuniform fiber diameter and spacing.

The analysis presented may be applied to longitudinal bending of plates, rather than beams, by substituting the following quantity for the longitudinal Young's modulus:

$$E/(1-\nu_{LT}\nu_{TL})$$

where ν_{LT} and ν_{TL} are the major and minor Poisson's ratios.

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NOMENCLATURE

a = area (general)
a_f, a_m = cross-sectional areas of fiber and matrix
a_{of}, a_{om} = thickness shear areas of fiber and matrix (see Figure 3)
a_o = $a + a$
a_{sf}, a_{sm} = horizontal shear areas of fiber and matrix (see Figure 2)
$\bar{a}$ = stretching stiffness of layer
A = cross-sectional area of laminated composite
b_f, b_m = widths of fiber and matrix at distance z
c = h/2
d = filament diameter
$\bar{d}, D$ = flexural rigidities of layer and of laminated composite
E = longitudinal Young's modulus
F, F_1, F_2 = shear strain-energy parameters, defined by Equations 30 and 31
F_h = horizontal shear force
G_f, G_m = shear modulus of fiber and matrix
h = thickness of layer
m = bending moment acting on layer
n = number of layers in multilayer composite
q = thickness-shear force on layer
r = d/2
s = thickness-shear flexibility, defined by Equation 34
s_h = thickness-shear flexibility of homogeneous beam
V_f = fiber volume fraction
t = time
U_b, U_s = strain energies due to bending and shear
w = beam deflection
W = $\bar{W}/2$
$\bar{W}$ = horizontal center-to-center distance between fibers
Y = shear factor, defined in Equation 19
x = position along beam
z = distance in thickness direction, measured from middle surface
α = rotation
β = factor defined in Equation 38
γ = shear strain
δ = variational symbol

TABLE 2. *Constituent-material properties and geometric parameters used in calculating shear flexibility efficiencies.*

Material	E, psi x 10^{-6}	G, psi x 10^{-6}	Ref.
Filament Materials:			
Boron	60.0	25.0	4
S-glass	12.0	4.85	17
Matrix Materials:			
Epoxy	0.5	0.185	4
Aluminum Alloy	10.0	3.8	—

Geometric Parameters:

$$h/\bar{W} = 0.85$$

$$\rho = d/h = 0.85$$

η_D	=	efficiency factor for flexural rigidity
η_s	=	efficiency factor for shear flexibility, Equation 37
ν_{LT}, ν_{TL}	=	major and minor Poisson's ratios
ρ	=	d/h
σ	=	bending stress
τ	=	shear stress
ϕ_b, ϕ_s	=	Reissner functional for bending and thickness shear

Subscripts:

f	=	filament
i	=	general subscript denoting f and m in general
k	=	arbitrary layer of multilayer beam
m	=	matrix
,x	=	differentiation with respect to x

TABLE 3. *Shear flexibility efficiencies.*

Composite	η_s
Boron/epoxy	236.
S-glass/epoxy	42.5
Boron/aluminum	0.885

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