The Use of the Calculus of Variations in Computing Production Schedules

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Aggregate production and employment scheduling has consistently been one of the most difficult problems in the area of production and operations management. This problem is concerned with determining what production rate and work-force level should be set at time t in order to minimize some criterion function associated with producing a product over a planning horizon of length T.

The United States Post Office Department is a prime example of an organization where manpower fluctuates on a continuous basis. Figure 1 is an illustration of the usual pattern of mail arrival in any major post office. $\lambda(t)$ is the amount of mail arriving at time t.

Variational methods were first used in analyzing post office problems by Oliver and Samuel (1961). Their criterion was the minimization of time delays as mail passes through various sorting stages. They came up with some surprising results by analysis of conditions for a solution.

During the Summer of 1968, I analyzed some single-stage problems where the criterion was minimization of cost. The following assumptions characterize some fundamental post office problems.

Let $x(t)$ be the amount of mail processed at time t. Let the integral from a to b be represented by $I(a, b)$. Then the assumptions are:

(1) $\lambda(t) = x(t)$

(2) $|x'(t)| \leq m$, a finite constant

(3) $I(0,T)\ (\lambda(t)\ dt) = I(0,T)\ (x(t)\ dt) = K$, a constant

(4) $x(t), \lambda(t) \geq 0$

(5) $\lambda(0) = 0, \lambda(T) = 0$

These assumptions state that the Post Office must meet demand as it occurs, total mail must be processed and is equal to K, production changes cannot be infinite, and the input rate rises from 0 and then returns to 0 as shown by Figure 1. The processing of third class mail roughly fits these assumptions.

Two elementary cost functions $c(t)$ are analyzed:

Case I $\quad c(t) = k_1x + k_2x'^2$

Case II $\quad c(t) = k_1x + k_2|x'|$

where x' is $[dx(t)]/dt$.

In Case I the problem is to minimize $I[x(t)] = I(0,T)\ (k_1x + k_2x'^2)\ dt$ subject to $I(0,T)\ (x(t)\ dt) = I(0,T)\ (\lambda(t)dt) = K$

This is a calculus of variations problem. The constraint forces the use of the Lagrange multiplier and hence the new total-cost function, taking into account constraint (3), is:

$$F(x, x', t, \lambda) = k_1x + k_2x'^2 + \lambda x \quad (6)$$

Since $I(0,T)\ [x(t)dt] = K$, λ is independent of t.

Application of the Euler condition to $F(x, x', t, \lambda)$ with $x(0) = x(T) = 0$ gives the solution.

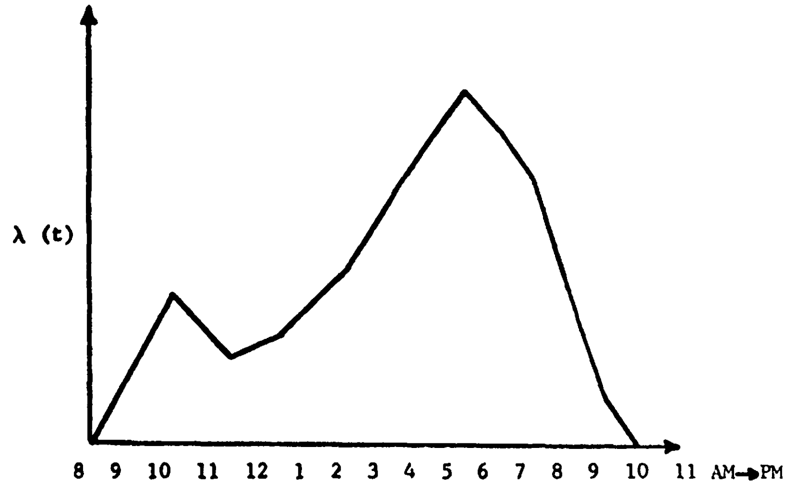

Figure 1. The General Pattern of Mail Arrival.

The Euler equation $\partial F/\partial x = d/dt\ (\partial F/\partial x')$ gives $k_1 + \lambda = 2k_2x'$ (7)

The solution form from the above equation (7) subject to the constraints is $x(t) = (6K/T^2)\ [t - (t^2/T)]\ 0 \leq t \leq T$ (8)

The sufficient condition for optimality is for $(\partial^2F/\partial x'^2) > 0\ \forall t,\ t \epsilon\ [0,T]$. Since $(\partial^2F/\partial x'^2) = 2$ this condition is satisfied.

The usefulness of this solution is indicated by the fact that $x(t)$ is independent of k_1 and k_2 which in practice are difficult and expensive to determine. The production schedule is a parabola. The schedule clearly anticipates the phase-out of production.

In Case II the minimization of $I = \int_0^T (k_1x + k_2\,|x'|)\ dt$ is sought.

Since constant schedules are allowed $x' = 0$ is feasible. However, $\partial F/\partial x'$ does not exist at $x' = 0$ and the Euler condition does not apply.

To minimize I it is sufficient to minimize $I(x') = \int_0^T |x'|\ dt$. This variational problem must be approached by new methods. For a physical situation,

since $|x'| = x' \qquad x' \geq 0$

$|x'| = -x' \qquad x' < 0$

the integral $I(x')$ can be written as a sum of intervals over which x' can have only a finite number of sign changes in the interval $[0,T]$.

Let the integral change sign i times in the interval $[0,T]$. Let $t_0 = 0$, $t_{i+1} = T$ and let t_n represent a point where x' changes sign where $n = 1, 2, ..., i$. Hence:

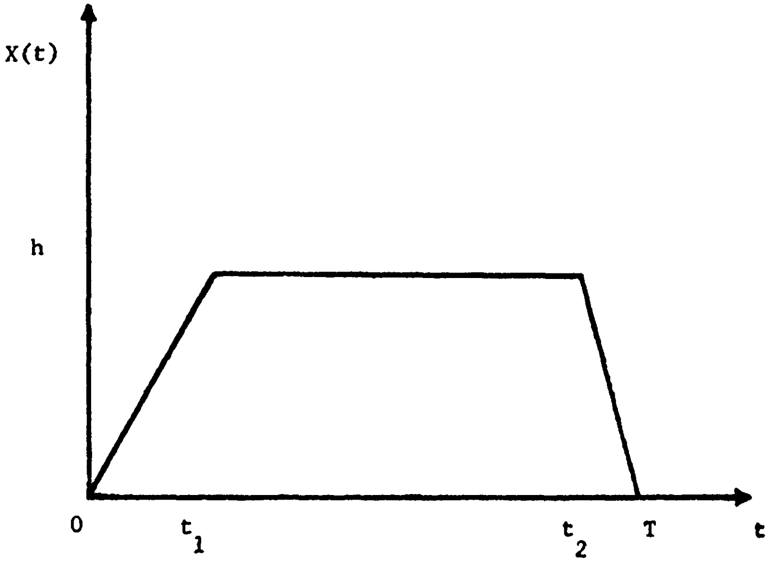

Figure 2. A Feasible Production Schedule.

$$I(0,T)\ |x'|\ dt = I(t_1,t_0)\ (x'dt) + I(t_2,t_1)\ (-\ x'dt) + I(t_3,t_2)\ (x'dt) + \ldots\ I(t_i,t_{i-1})\ (x'dt) + I(t_{i+1},t_i)\ (-\ x'dt)$$

The last sign must be negative since production phases to zero.

Then: $I(0,T)\ |x'|\ dt = x(t_1) - x(t_0) - x(t_2) + x(t_1) + x(t_3) - x(t_2)$

$$+ \ldots + x(t_i) - x(t_{i-1}) - x(t_{i+1}) + x(t_i) = 2 \sum_{n=1}^{i} x(t_n)\ (-1)^{n+1} \quad (9)$$

From (9) it can be concluded that the value of $I(x')$ and hence $I(x)$ depends only on the points where the curve changes sign and the ordinates of the curve at these points. Therefore there may be infinitely many minimizing curves. Hence straight line patterns may be analyzed and still retain generality.

If the schedule in Figure 2 is analyzed then $I(0,T)\ |x'|\ dt$ subject to $I(0,T)xdt = K$ is $4K/(T + t_2 - t_1)$

The integral will be a minimum if $t_2 - t_1$ is allowed to approach T. Then the value of the integral is $4K/(T + T) = 2K/T$ and $x(t) = K/T$ = a constant.

This verifies the usual production rule to build up production instantaneously and then maintain a constant production rate.

These solutions indicate that arrival rate, subject to (5), should be controlled in a different pattern from Figure 1, if (1) is to be satisfied and cost minimized.

Reference Cited

Oliver, R. L. and A. H. Samuel. 1961. Reducing letter delays in post offices. Res. Rep. 1, Operations Research Center, Univ. Calif., Berkeley.