Effect of Curvature on the Temperature Profiles in Conducting Spines

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INTRODUCTION

The problem considered in this paper is the determination of the effect of surface curvature on the heat flux and temperature distribution in a spine or pin fin. For one-dimensional, heat-transfer calculations Schneider (1955) and Jakob (1949) indicate that it is normal to assume that the geometry of the spine is such that the length measured along the surface is the same as the length measured along the axis of symmetry. This implies that the slope of the surface is small. The present work includes the effect of a non-zero slope and presents the differences obtained in predicting heat flux and temperature distribution. Three spines have been considered, each having a circular cross section, with the radius varying with position along the spine axis as indicated in Table I. The position along the spine axis is measured from the tip, and "a" is a constant.

TABLE I. SPINE GEOMETRY

Profile	Equation of Radius
1	radius $= ax = y$
2	radius $= ax^2 = y$
3	radius $= ax^{1/2} = y$

ANALYSIS

It is assumed that the spine has constant thermal conductivity, and that the convective heat transfer coefficient of the fluid surrounding the spine is also constant. One-dimensional heat transfer is considered so that the length of the spine must be large compared to its thickness.

Figure 1 illustrates the coordinate system for this problem.

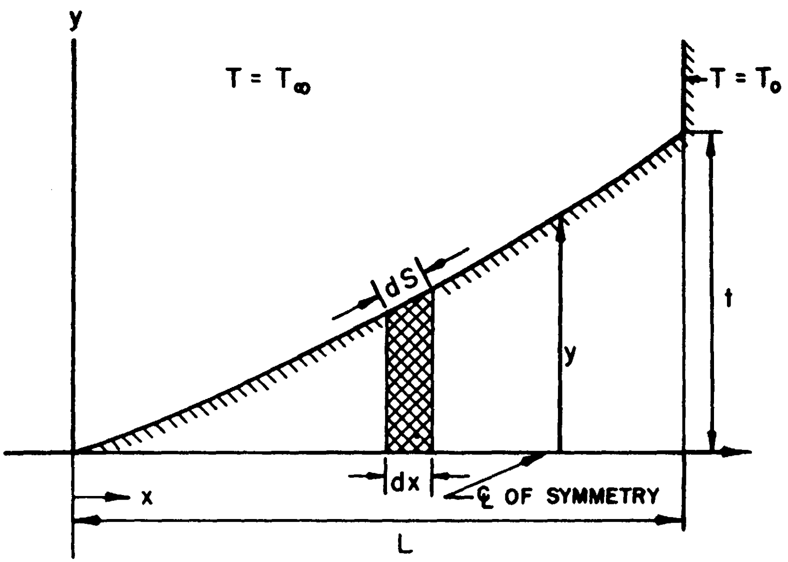

Figure 1. Coordinate System

The governing differential equation is obtained by applying an energy balance to a differential volume as shown in Figure 1.

Conservation of energy yields

$$(d^2T/dx^2) + (dT/dx)\,[(1/A)\,(dA/dx)] - [(h/k)\,(P/A)\,(dS/dx)\,(T - T_\infty)] = 0. \qquad (1)$$

where k = thermal conductivity, h = convective coefficient, P = perimeter, and A = cross-sectional area. For a fin of circular cross section, a spine, whose surface is described by an equation of the form $y = ax^n$, equation (1) becomes

$$d^2T/dx^2) + (2n/x)\,(dT/dx) - (h/k)\,(2/ax^n)\,[1 + (anx^{n-1})^2]^{1/2}\,(T - T_x) = 0. \qquad (2)$$

The following substitution of variables will allow the conservation equation to be written in dimensionless form:

$$\Phi = (T - T_\infty/T_o - T_\infty),\ \xi = x/L,\ \beta = hL/k, \text{ and } \alpha = t/L.$$

The result of this transformation of equation (2) to dimensionless form is

$$\Phi'' + (2n\Phi'/\xi) - (2\,\beta/\alpha)\,(1/\xi^n)\,[1 + (\alpha\, n\, \xi^{n-1})^2]^{1/2}\,\Phi = 0. \qquad (3)$$

The boundary conditions which must be satisfied are as follows:

at $x = 1$, $T = T_o$ or for equation (3) at $\xi = 1$, $\Phi = 1$,

at $x = 0$, $dT/dx = 0$ or for equation (3) at $\xi = 0$, $d\Phi/d\xi = 0$.

Equation (3) reduces to the generalized Bessel's equation if the second term in the radical is assumed small compared to unity. This assumption is analogous to assuming *ds* is equivalent to *dx*, (see Figure 1). A closed form solution to equation (3) has not been found, and the solutions presented here were obtained by the application of the fourth-order Runge-Kutta method (McCracken and Dorn, 1964).

Results

The results of this study are given in three illustrations. Table II summarizes and compares the numerical values of dimensionless heat flux obtained through the use of the simplifying approximation on surface slope (Bessel's equation) with the values obtained from the solution of equation (3).

TABLE II. COMPARATIVE DIMENSIONLESS HEAT FLUX

Dimensionless Heat Flux $= QL/kA(T_o - T_\infty)$		
Profile	Bessel	This Paper
1	1.542	1.564
2	1.192	1.241
3	1.808	1.814

Figure 2 presents the temperature distribution for the various spine profiles given in Table I. Figure 3 presents the per cent error between the temperatures obtained from the solution of equation (3) and the temperatures obtained from the solution of Bessel's equation.

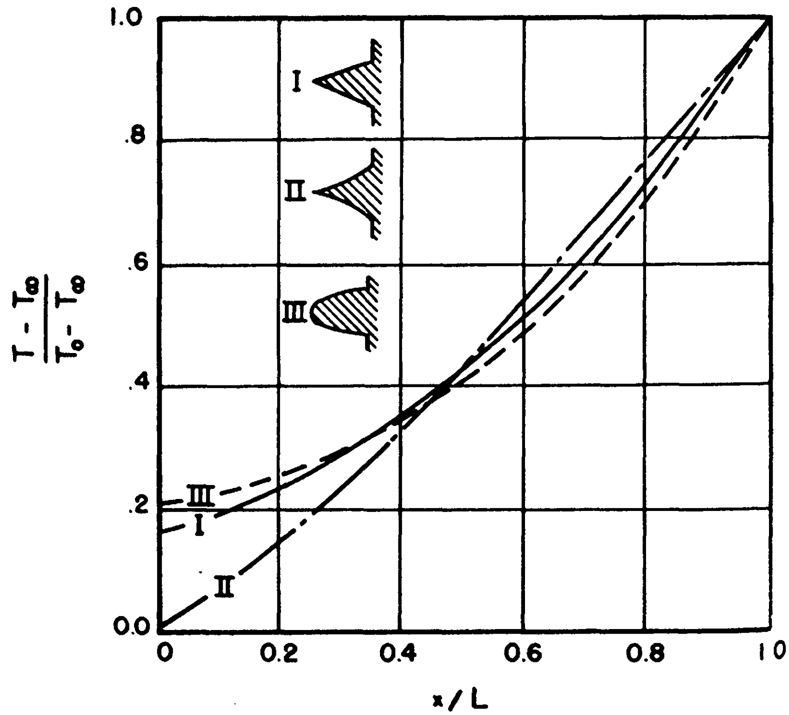

Figure 2. Dimensionless Temperature Profiles

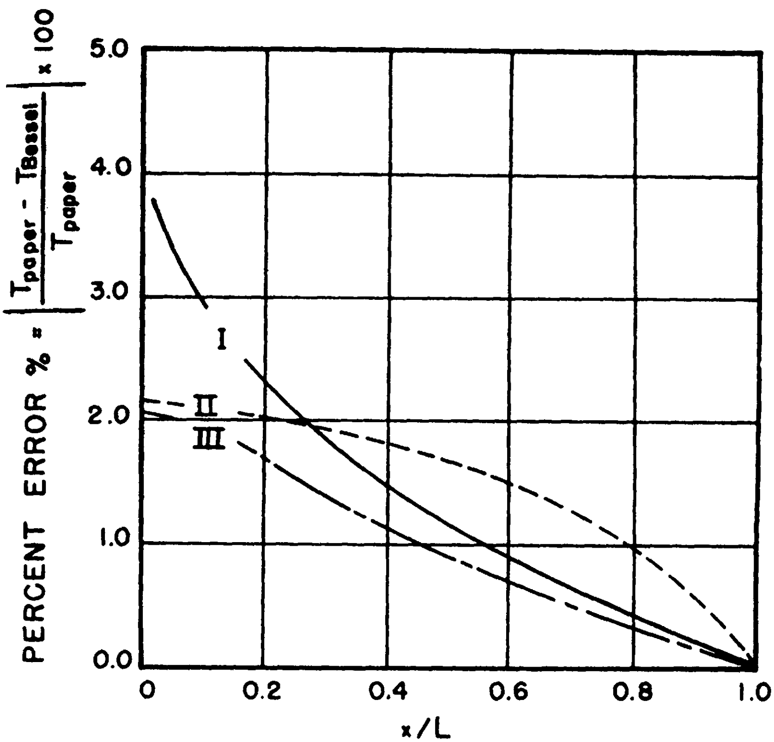

Figure 3. Percentage Error Between Exact and Approximate Solutions

CONCLUSIONS

The more careful analysis presented in this paper does produce results which are more accurate than those of the approximate method normally used. In the case of temperature distribution the error involved is significant though not so great as to preclude the use of the approximate method for some engineering problems. In the case of heat flux the tabulated values indicate a difference between the approximate method and that of this paper which is large enough to be of some concern, particularly in the case of greater surface curvature. The nature of the Bessel's solution is often such that no tabulated values of the appropriate Bessel's function are available to implement the equation, and for this reason the numerical solution indicated in this paper is more satisfactory for individuals having access to computers.

The methods used in this work can be extended to handle other surface shapes as well as straight and circular fins of variable cross section. Further study is necessary to determine if the error introduced by the one-dimensional assumption is of the same order of magnitude as the error introduced by neglecting the curvature of the surface.

Commentary by B. L. Dowty, consultant. — I . . . . believe that the authors have failed to recognize one important fact which makes their work rather meaningless. The assumption of fin length very much greater than fin base, i.e. $L >> b$, which is required for the one-dimensional analysis, implies ds of the surface is approximated by dx from: $ds = [1 + (dy/dx)^2]^{1/2}\ dx$ where $dy/dx = 0\ (b/L)$. Thus, the "corrected analysis" the authors perform cannot improve the results since it does not relax any of the assumed restrictions.

Author's reply.—We wish to thank Mr. E. L. Dowty for his comments. The assumption of one-dimensional heat transfer which implies that the fin length is much greater than the fin base does not necessarily imply that ds of the surface is nearly equal to the length dx along the axis at all values of x. For example, in case III of the paper, near the spine tip $ds >> dx$. The intent of the present paper was to investigate the temperature profile and heat flux which are obtained when one assumes that ds is not equal to dx. The differences between this case and the case when ds is equal to dx are admittedly small, but in some applications such as thermocouple probes, they may be significant. In general, for the spine shape given by $y = ax^n$ or $y = ax^{1/n}$, the effect of curvature should become more important with larger values of n.

REFERENCES

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