SECTION I, ENGINEERING SCIENCE

On the Application of Dimensional Analysis

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Many problems facing engineers today are of such complexity that a direct mathematical solution becomes very difficult or even unattainable. For example, the answer to a seemingly simple engineering problem may involve solving several nonlinear partial differential equations with nonhomogeneous and/or nonlinear boundary conditions. In a select number of instances, the use of similarity transformations permits reducing the describing equations to a set of ordinary differential equations. This greatly helps implement a solution, even though the resultant equations may very well be nonlinear. Ordinary differential equations which possess nonlinearities are more amenable to analytical solution than nonlinear partial differential equations. When a problem is so difficult that an accurate mathematical solution is, at least, temporarily abandoned, carefully designed experiments are normally carried out to obtain the required solution. To properly design such an experiment, a complete knowledge of all pertinent parameters needing investigation is required.

The purpose of this paper is to present a rapid and straightforward method of applying dimensional analysis to engineering problems that pose difficulties such as those outlined in the above paragraph, and which can be represented by a mathematical model. The procedure to be described developed from an article published several years ago (Hellums and Churchill, 1961). In short, it will be demonstrated how the minimum number of significant dimensionless parameters describing a problem can be determined. Further, the method will show how the functional relationships describing a problem are generated, and how appropriate changes of variable are formulated for analytically solving certain problems.

Summary of Method

The application of dimensional analysis to mathematically represent engineering problems consists of several steps. These are: (1) Formulate all differential equations and/or algebraic expressions which adequately describe the problem, together with the required boundary conditions; (2) select the appropriate dimensionless variables for all independent and dependent variables involved, using arbitrary terms or boundary values for the denominator in each case; (3) substitute the new dimensionless variables into each differential equation and boundary condition to normalize them; (4) group into functional form all dimensionless variables and those parameters generated; (5) reduce the functionality to the minimum possible number of independent groups; and (6) drop any groups in which the arbitrary terms can not be divided out using other groups of the functionality. The following example is used to demonstrate and exemplify the ramifications of the method.

Mass Transfer in a Falling Liquid Film—Consideration is given to a liquid film of constant thickness, h, falling in laminar flow down a vertical wall, as shown in Figure 1. The liquid is originally pure B, but, at some point along the wall, component A begins diffusing through the wall into the liquid at a constant rate, N_A.

Continuity equation for component A is

$$(1) \quad u\, C_z = D_{AB}\, C_{yy} \quad y \geq 0,\ z \geq 0$$

[1]Partial support received from the National Science Foundation (Research Grant Gi-1028).

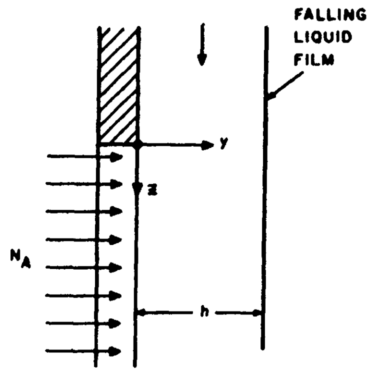

Fig. 1. Falling liquid film

where u is the downward liquid velocity, C the concentration of component A, C_z and C_{yy} the first and second partial derivatives with respect to z and y, and D_{AB} the mass diffusivity of component A. Appropriate boundary conditions for the problem are:

$$(2) \quad C = 0 \qquad z \leq 0 \quad y > 0$$

$$(3) \quad C = 0 \qquad \text{all } z \quad y \rightarrow \infty$$

$$(4) \quad C_y = -N_A/D_{AB} \qquad z > 0 \quad y = 0$$

Near the wall, the velocity distribution may be approximated by the linear function (Bird et al., 1960)

$$(5) \quad u_z = (\rho g h/\mu) y$$

where ρ and μ are the liquid density and viscosity and g the acceleration of gravity. From Eq. (5), and letting

$$\lambda = \mu D_{AB}/\rho g h,$$

Eq. (1) becomes

$$(6) \quad y\, C_z = \lambda\, C_{yy}$$

The dimensionless variables $C' = C/C_o$, $Z' = z/z_o$ and $Y = y/y_o$ are selected, where C_o, z_o and y_o are arbitrary. Substitution of these variables into Eq. (6) and the boundary conditions, Eqs. (2) — (4), and normalizing yields

$$(7) \quad YC'_{Z'} = [\lambda z_o/y_o^3] C'_{YY}$$

and

$$(8) \quad C' = 0 \qquad Z' = 0 \quad Y > 0$$

$$(9) \quad C' = 0 \qquad \text{all } Z' \quad Y \rightarrow \infty$$

$$(10) \quad C' = (-N_A y_o)/(D_{AB} C_o) \qquad Z' > 0 \quad Y = 0$$

Note that the boundary condition represented by Eq. (10) remains nonhomogeneous; thus it must be retained as a parameter. Grouping all independent parameters in functional form gives

$$(11) \quad f\,[C/C_o,\ z/z_o,\ y/y_o,\ y_o^3/\lambda z_o,\ C_o D_{AB}/N_A y_o] = 0$$

At this point in the analysis, it is generally a good policy to examine individual groups to see if they are indeed dimensionless. Also, it should be mentioned that dropping signs on groups, inverting groups, multiplying or dividing groups by one another or changing the power on any group does not affect the functionality, i.e., $y^3_o \lambda z_o$ could be represented equally well as $y/(\lambda z_o)^{1/3}$ or $(\lambda z_o)^{1/3}/y$.

To minimize the number of groups in Eq. (11) that contain the arbitrary terms C_o, z_o, y_o requires: (1) Multiplying the fifth group by the first group over the third group, and (2) multiplying the fourth group by the cube of the third group divided by the second group. The result is

$$(12) \quad f\,[C/C_o,\ z/z_o,\ y/y_o,\ y^3/\lambda z,\ CD_{AB}/N_A y] = 0$$

Now, taking the cube root of the fourth group gives

$$(13) \quad f\,[C/C_o,\ z/z_o,\ y/y_o,\ y/(\lambda z)^{1/3},\ CD_{AB}/N_A\ y] = 0$$

or in an equivalent form

$$(14) \quad (C/N_A\ y) D_{AB} = f\,[z/z_o,\ y/y_o,\ y/(\lambda z)^{1/3}]$$

The final expression representing the minimum number of dimensionless parameters describing the problem is obtained by dropping out the parameters in Eq. (14) still containing the arbitrary terms C_o, z_o and y_o. Consequently,

$$(15) \quad CD_{AB}/N_A\ y = f\,[y/(\lambda z)^{1/3}]$$

An equivalent form is

$$(16) \quad CD_{AB}/N_A(\lambda z)^{1/3} = f\,[y/(\lambda z)^{1/3}]$$

Thus, the minimum number of significant parameters describing the problem is represented by either Eq. (15) or Eq. (16).

To determine the parameters affecting the concentration of A at the wall, simply let $y = 0$, giving

$$(17) \quad CD_{AB}/N_A(\lambda z)^{1/3} = \text{const}$$

or

$$(18) \quad C \propto z^{1/3}$$

This useful result, determined solely by dimensional analysis, shows that the wall concentration of A varies explicitly as the cube root of vertical distance along the wall.

To obtain an analytical solution requires letting

$$(19) \quad CD_{AB}/N_A\ y = f(x)$$

where $x = y/(9\lambda z)^{1/3}$. The $(9)^{1/3}$ in the expression is placed there only for later convenience, and does not affect the applicability of the solution method.

Substitution of the similarity transformation, Eq. (19), into Eq. (6) gives the ordinary differential equation

$$(20) \quad f'' + [(2/x) + 3x^2]\,f' = 0$$

where primes represent derivatives of $f(x)$ with respect to the variable x. Integrating twice and applying the boundary conditions shown by Eqs. (2) — (4) gives

$$(21) \quad f = CD_{AB}/N_A\ y = [1/\Gamma(2/3)] \int_x^\infty (e^{-x^3}/x^2)\,dx$$

where Γ (⅔) is a gamma function. This represents the complete analytical solution to the problem.

To summarize, a procedure has been demonstrated for determining the minimum number of significant parameters describing an engineering problem which can be represented by a mathematical model. For the particular example chosen, derived functional relationships were utilized as similarity transformations to aid in carrying out the analytical solution.

LITERATURE CITED

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