Applicability of Aitken's δ^2 Process in Numerical Integration

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The remainders of Newton-Cotes and various other simplex numerical quadrature formulas are of the form

$$R = c(b-a)\ (\Delta x)^n f^{(n)}(\xi),\ a \leq \xi \leq b$$

where a and b are, respectively, the lower and upper limits of integration, c and n are constants whose values depend upon which numerical quadrature formula is being used, Δx is the (constant) width of the subintervals ("panels") of $[a, b]$, and ξ is an unknown value of the variable of integration lying somewhere between the lower and upper limits of integration.

It was shown by Richardson (Richardson, 1910; Richardson and Gaunt, 1927) that if the same numerical quadrature formula is used with different widths of subintervals, $(\Delta x)_1$, and $(\Delta x)_2$, the resulting estimates of the integrals I_1 and I_2 respectively, can be improved by "h^n- extrapolation":

$$I \simeq I_2 + (I_2 - I_1)/\{[(\Delta x)_1/(\Delta x)_2]^n - 1\}$$

Since, for a given interval of integration, $[a, b]$, the width of the subintervals is inversely proportional to the number of subintervals into which $[a,b]$ is subdivided, we can replace $[(\Delta x)_1/(\Delta x)_2]^n$ by the more convenient ratio $(P_2/P_1)^n$, where P_i denotes the number of subintervals. (Richardson's formula is ordinarily not written this way even though it would be more convenient to do so.)

We shall now show that Aitken's "δ^2 extrapolation" (Aitken, 1937) can also be used to obtain improved estimates of the values of integrals under the proper conditions.

Writing ρ for the ratio P_2/P_1, Richardson's formula is

$$I \simeq I_2 + (I_2 - I_1)/\ (\rho - 1)$$

and can be rearranged to

$$(I - I_2)\ /\ (I_2 - I_1) \simeq 1/\ (\rho - 1)$$

Now suppose that the same ratio, ρ, holds for the estimates I_3 and I_2 as for I_2 and I_1. Then

$$(I - I_2) / (I_2 - I_1) \simeq (I - I_3) / (I_3 - I_2)$$

Solving for I, we find that

$$I \simeq (I_1 I_3 - I_2^2) / (I_1 + I_3 - 2I_2)$$

which is of the same form as Aitken's δ^2-extrapolation formula. According to the above derivation, δ^2 extrapolation is applicable to numerical quadrature when

$$(\Delta x)_1 / (\Delta x)_2 = (\Delta x)_2 / (\Delta x)_3$$

or

$$(\Delta x)_1 : (\Delta x)_2 : (\Delta x)_3 :: r^2 : r : 1$$

It is especially convenient in carrying out computations to let $r = 2$.

Whereas the form of Richardson's formula depends on the order of the derivative in the remainder term but does not require any particular ratio of subinterval widths, Aitken's formula applied to numerical quadrature does not depend on the order of the derivative but does necessitate the above ratios of subinterval widths.

By way of example, we shall estimate

$$\ln 10 = \int_1^{10} dx/x = 2.3025850$$

using the midpoint numerical quadrature formula. The results are as follows:

Number of Subintervals	Midpoint Area	h^n Extrapolation	δ^2 Extrapolation
1	1.6363636		
2	1.9652606	2.0748020	
4	2.1662530	2.2332505	2.4820973
8	2.2581667	2.2888046	2.3356165
16	2.2901689	2.3008363	2.3072631
32	2.2993656	2.3024312	2.3030743
64	2.3017690	2.3025701	2.3026193
128	2.3023771	2.3025798	2.3025812 (best)
256	2.3025299	2.3025808	2.3025812

Sometimes h^n extrapolation is more accurate, while at other times δ^2 extrapolation is. It should be noted also that extrapolation does not always result in improvement, although in the present example it did so with one exception (the last δ^2 extrapolation). These observations should provide a useful basis for further research.

LITERATURE CITED

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