Acceleration of the Convergence of the Secant Method

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In the secant method for finding zeros of functions, for which the iteration formula is

$$x_{n+1} = x_n - [(x_n - x_{n-1})f(x_n)]/[f(x_n) - f\ (x_{n-1})]$$

the errors of the successive iterates x_0, x_1, x_2, . . . , defined by

$$e_i = x_i - z\ ,\ i = 0, 1, 2, \ldots$$

where z is the zero sought, are related by the error formula

$$e_{n+1} = [f''(\xi_{n-1})/2f'(\xi_{n-1})]\ e_n e_{n-1},\ \xi_{n-1} \in (x_{n-1}, z)$$

The error of x_n is then given by

$$e_n = [f''(\xi_{n-2})/2f'(\xi_{n-2})]\ e_{n-1}e_{n-2},\ \xi_{n-2} \in (x_{n-2},\ z)$$

Assuming that $f''(\xi_{n-1})\ /2f'(\xi_{n-1})$ equals $f''(\xi_{n-2})\ /2f'(\xi_{n-2})$ sufficiently closely, we can eliminate these factors between the two error formulas, obtaining

$$\varepsilon_{n+1}/\varepsilon_n \simeq \varepsilon_n\varepsilon_{n-1}/\varepsilon_{n-1}\varepsilon_{n-2} = \varepsilon_n/\varepsilon_{n-2}$$

or

$$(x_{n+1} - z)/(x_n - z) \simeq (x_n - z)/(x_{n-2} - z)$$

whence

$$z \simeq (x_{n+1}\, x_{n-2} - x_n^2)/(x_{n+1} - 2x_n + x_{n-2})$$

This result, which provides an improved estimate of z, is the same as Aitken's δ^2 formula except that x_{n-1} in Aitken's formula is here replaced by x_{n-2}; thus, *four* successive iterates are needed instead of three.

Since no conditions were placed on the factors that were eliminated other than that they be sufficiently nearly equal, their magnitudes can exceed unity, whence the above formula for accelerating the convergence of the secant method is also applicable to divergent sequences, forcing them to converge.

As the order of the secant method is 1.618, the use of the secant method in conjunction with the above convergence acceleration formula provides a very rapidly converging, yet simple, procedure for finding zeros of functions.