The Near Field of a Horizontal Finite Electric Dipole at the Surface of an Anisotropic Earth

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An analytical study is in progress to determine the optimum method to employ electrodynamic fields for ascertaining the geological features which are under the surface of the earth. This paper presents the analytically predicted fields at the surface of the earth which result from power input to a finite electric dipole at the surface of an anisotropic earth. The solution in this paper is only one of nearly a dozen that were obtained to investigate different techniques for applying electric and magnetic fields to an earth (a) with homogeneous layers, or (b) with anisotropic conductivity. In this problem, an electric potential is applied to two electrodes with a separation of 100 m (a dipole of finite length) at the surface of the earth. The electrical resistivity of the earth is assumed to be anisotropic with a horizontal resistivity of 50 ohm-meters and a vertical resistiviy of 5000 ohm-meters. A detector measures the electrodynamic fields on the surface at various distances from one electrode of the source, as is indicated in Figure 1A. The receiving stations are not on a straight line, but are included in an angular wedge between 30 and 45 degrees with the axis of the dipole, i.e., the line through the two electrodes. The receiver measures the magnetic and the electric fields in the direction x, which is parallel to the axis of the dipole, and in the direction y, which is perpendicular to the axis.

With the aid of a large digital computer, the solution for a finite length of dipole may be obtained from the solution for an infinitesimal dipole. A series of infinitesimal dipoles is extended along a straight line with the charge at one end of each dipole superimposed on the opposite charge of the preceding infinitesimal dipole. This technique is illustrated in Figure 1B. The process may be continued to obtain any desired finite length of dipole. This paper is concerned with finding the fields for an infinitesimal dipole. There are numerous, approximate solutions for the problem of infinitesimal dipoles at the surface of an isotropic earth (Wait, 1953; Rikitake, 1966). In this paper, an exact solution (to the extent that an integral may be evaluated) is obtained for an infinitesimal dipole at the surface of an anisotropic earth. The anisotropic solution reduces easily to the solution for an isotropic earth.

Initial and Boundary Conditions

The solution for the magnetic field vector, H, and the electric field vector, E, that are established by an infinitesimal dipole may be evaluated in terms of the Sommerfeld and the Fok integrals. In order to reduce the complexity of these integrals, the initial and boundary conditions may be defined to simplify the solution without overly restricting its generality. The simplification is obtained by restricting the frequency range from a low of roughly 10 sec/cycle to a high of perhaps 100 to 1000 cycles/sec. The frequency affects the depth of penetration of the conduction current into the earth by the familiar skin effect. With these frequencies, finite electric dipoles are very short when compared to the wavelength, such as 100 to about 10^7 meters, or longer. With this ratio, potential theory applies to the practical problem.

For a sinusoidal input to the dipole, Maxwell's equations may be replaced by their Fourier transforms which include the harmonic depend-

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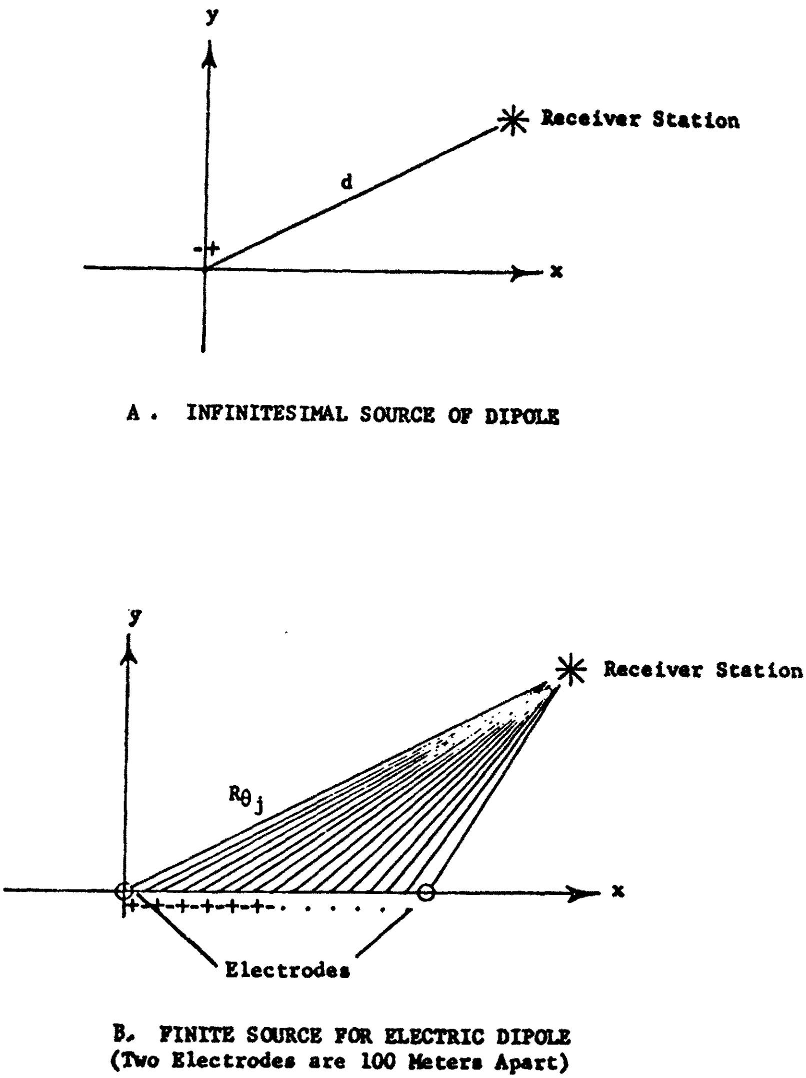

Figure 1. The surface arrangement of an infinitesimal and a finite dipole with respect to the receiver.

ency, $e^{i\omega t}$. The term ω is equal to $2\pi f$, where f is the frequency. A non-sinusoidal pulse of current to the dipole will require the several frequency components which are found from a Fourier series reconstruction of the pulse.

The highest frequency is so small that displacement currents (capacitive currents) may be neglected in comparison to the conduction currents. Although some rocks may have appreciable permeability, it may be closely approximated by the permeability of free space. In potential theory, the two preceding statements are equivalent to stating that the permeability, μ_o, and the permittivity, ε_o, of free space may be used.

Ground currents are induced in the earth from the occurrence of ionization in the upper atmosphere and from motion of this ionized region. Random ground currents, with various directions and frequencies, are called magneto-telluric currents. Their effect may be eliminated from the measurements. A detector is employed that measures the fields at the input frequency of the power to the finite length of dipole and then averages the reading over several cycles.

The boundary conditions assume a semi-infinite earth with a flat surface in the xy-plane which is at $z = 0$. For negative values of z, the air is assumed to be nonconducting and to have no effect on the solution for the earth. For positive values of z, the earth is assumed to have anisotropic conductivity. The components of the conductivity tensor are defined by the following relations.

$$\sigma_{xx} = \sigma_{yy} = \sigma_t$$

$$\sigma_{zz} = \sigma_n$$

$$\sigma_{ij} = 0 \quad \text{when } i \neq j$$

The definition states that the traverse conductivity, σ_t, is different from the conductivity normal to the surface of the earth, σ_n. The usual sedimentary deposit has been observed to have $\sigma_t < \sigma_n$

Form of the Solution

The Fourier transforms of the Maxwell equations may be written in the form (Jackson, 1962)

$$\operatorname{div} \vec{B}_\omega = 0 \; ; \quad \operatorname{curl} \vec{E}_\omega + i\omega\vec{B}_\omega = 0$$

$$\operatorname{div} \vec{E}_\omega = \frac{1}{\tilde{\tilde{\sigma}}_\omega \epsilon_o} \; ; \quad \operatorname{curl} \vec{B}_\omega - \mu_o \tilde{\tilde{\sigma}}_\omega \vec{E}_\omega = 0 \qquad (1)$$

$\tilde{\tilde{\sigma}}$ is the conductivity tensor.

The displacement current is omitted from the equation for curl B since such current is assumed to be negligible. The relations between the fields, the potential vector, and the scalar potential, ϕ, are

$$\vec{B} = \operatorname{curl} \vec{A}$$

$$\vec{E} = i\omega\vec{A} + \nabla\phi = 0 \qquad (2)$$

$\vec{A}$ is the magnetic vector potential.

Combining Equations (1), (2), the initial boundary conditions and the restrictions on the conductivity tensor yields the differential equations for the components of A in the following form:

$$\nabla^2 A_x - i\omega\mu\sigma_t A_x = 0$$

$$\frac{\delta^2 A_z}{\delta x^2} + \frac{\delta^2 A_z}{\delta y^2} + \frac{\sigma_n}{\sigma_t}\frac{\delta^2 A_z}{\delta z^2} = i\omega\mu\sigma_n A_z + \left(1 - \frac{\sigma_n}{\sigma_t}\right)\frac{\delta^2 A_x}{\delta x \delta z} = 0 \qquad (3)$$

A solution is to be obtained for the differential Equations 3 with appropriate boundary conditions. For this particular problem, it is convenient to obtain the solution as a function of cylindrical coordinates (r, θ, z). The most general solution for Equations (3) was given by Van'yan (1963).

$$A_z = \frac{\mu I l}{4\pi} \frac{\delta}{\delta x} \int_0^\infty J_o(\lambda r)\ G(Z,\lambda)\ \delta\lambda$$

$$A_x = \frac{\mu I l}{4\pi} \int_0^\infty J_o(\lambda r)\ F(Z,\lambda)\ \delta\lambda \tag{4}$$

in which the functions F (Z, λ), G (Z, λ) and Bessel functions, J_o, are to be evaluated by the use of the following boundary conditions. These boundary conditions follow from the requirement that the tangential components of E and H be continuous across the air-earth interface.

$$H_{x_o} = H_{x_1}$$

$$H_{y_o} = H_{y_1}$$

$$E_{x_o} = E_{x_1}$$

$$E_{y_o} = E_{y_1} \tag{5}$$

where the subscripts 0 and 1 represent air and earth, respectively. These relations are to be interpreted as illustrated by writing the first relation in words. The component of the magnetic field, H_x, in the earth at the air-earth interface is equal to the component, H_x, in the air at the same interface.

The two functions F and G may be evaluated from Equations 3 and 4 by applying the boundary conditions from Equation 5. The final form of these functions is given by the following relations

$$F_1(Z,\lambda) = \frac{2\lambda}{\lambda + \sqrt{\lambda^2+k^2}}\ e^{-Z\sqrt{\lambda^2+k^2}}$$

$$G_1(z,\lambda) = \frac{2}{\lambda}\left[\frac{\sqrt{\lambda^2+k^2}}{\lambda+\sqrt{\lambda^2+k^2}}\ e^{-z\sqrt{\lambda^2+k^2}} - e^{-z\sqrt{\lambda^2\frac{\sigma_1}{\sigma_n}+k^2}}\right] \tag{6}$$

The following functions are to be employed and are defined by these relations. Sommerfeld's integral (Sommerfeld, 1949) has the form

$$P = \int_0^\infty J_o(\lambda r)\ \frac{\lambda}{\sqrt{\lambda^2+k^2}}\ e^{-Z\sqrt{\lambda^2+k^2}}\ \delta\lambda = \frac{e^{-kR}}{R}$$

The Fok integral may be defined in terms of the modified Bessel functions, I_o and K_o, by the following relation

$$N = \int_0^\infty J_o(\lambda r)\ \frac{1}{\sqrt{\lambda^2+k^2}}\ e^{-Z\sqrt{\lambda^2+k^2}}\ \delta\lambda = I_o\left[\frac{k}{2}(R-z)\right] \cdot K_o\left[\frac{k}{2}(R-z)\right]$$

In the preceding relation, the modified Bessel functions are defined for any value of the coordinate z, which is the coordinate that measures the depth in the earth. Introduce another relation

$$P' = \frac{e^{-k\sqrt{z^2 + r^2 \frac{\sigma_n}{\sigma_t}}}}{\sqrt{z^2 + r^2 \frac{\sigma_n}{\sigma_t}}}$$

where $R = (x^2 + y^2 + z^2)^{1/2}$, $r = (x^2 + y^2)^{1/2}$ and $r' = [(x^2 + y^2)\ \sigma_n/\sigma_t]^{1/2}$. These functions and the functions from Equation 6 are substituted into the differential relation, Equation 3. The following general relations for the components of the electric and magnetic fields of the infinitesimal dipole are obtained. They are applicable for any position in the earth.

$$E_x = \frac{Il}{2\pi}\left\{ \frac{i\mu_0\omega}{k^2}\left(\frac{\delta^2 P}{\delta z^2} - \frac{\delta^3 N}{\delta z\,\delta y^2}\right) - \frac{1}{\sigma_t}\left[\frac{\delta^3 N}{\delta x^2 \delta z} - \frac{y^2 - x^2}{r^4}\cdot(P - P') - \frac{x}{r^2}\left(\frac{\delta P}{\delta x} - \frac{\delta P'}{\delta x}\right)\right]\right\} \qquad (7)$$

$$E_y = \frac{Il}{2\pi\sigma_t}\left[\frac{x}{r^2}\left(\frac{\delta P}{\delta y} - \frac{\delta P'}{\delta y}\right) - \frac{2xy}{r^4}(P - P') - \frac{\delta^3 N}{\delta x \delta y \delta z}\right] \qquad (8)$$

$$H_x = \frac{Il\,xy}{2\pi r^2}\frac{\delta}{\delta y}(P - P') - \frac{Il}{2\pi k^2}\left(\frac{\delta^3 P}{\delta x \delta y \delta z} + \frac{\delta^4 N}{\delta x \delta y \delta z^2}\right) \qquad (9)$$

$$H_y = \frac{Il}{2\pi k^2}\left(\frac{\delta^3 P}{\delta z^3} - k^2\frac{\delta^2 N}{\delta z^2} + \frac{\delta^4 N}{\delta z^4}\right) + \frac{Il}{2\pi k^2}\left(\frac{\delta^3 P}{\delta x^2 \delta z} + \frac{\delta^4 N}{\delta x^2 \delta z^2}\right)$$

$$- \frac{Il\,z}{2\pi}\frac{\delta}{\delta x}\left[\frac{x}{r^2}(P - P')\right] \qquad (10)$$

$$H_z = -\frac{Il}{2\pi k^2}\left(\frac{\delta^3 P}{\delta y \delta z^2} - k^2\frac{\delta^2 N}{\delta y \delta z} + \frac{\delta^4 N}{\delta y \delta z^3}\right) \qquad (11)$$

The numerical values, which may be easily measured in a program of geophysical prospecting, are the values of the above listed components at the surface of the earth. These components may be evaluated at the surface by setting $z = 0$ in all of the preceding general equations. This also requires the substitution of r for R. At the surface, the preceding equations are found to have the following form.

$$E_x = \frac{Il}{2\pi\sigma_t r^3} \left\{\left[\left(\frac{3x^2}{r^2} - 3\right) + e^{-kr}(1 + kr)\right]\right.$$

$$+ \left[\left(1 - \frac{3x^2}{r^2} - \frac{kx^2}{r}\right) e^{-kr}\right.$$

$$\left.\left. - \left(\sqrt{\frac{\sigma_n}{\sigma_t}} - \frac{3x^2}{r^2} \cdot \sqrt{\frac{\sigma_n}{\sigma_t}} - \frac{kx^2}{r}\right) e^{-kr\sqrt{\frac{\sigma_n}{\sigma_t}}}\right]\right\} \qquad (12)$$

$$E_y = \frac{3Ilxy}{2\pi\sigma_t r^5}\left\{1 + \left[\left(\sqrt{\frac{\sigma_n}{\sigma_t}} + \frac{kr}{3}\right)e^{-kr\sqrt{\frac{\sigma_n}{\sigma_t}}} - \left(1 + \frac{kr}{3}\right)e^{-kr}\right]\right\} \qquad (13)$$

$$H_x = -\frac{Ilxy^2}{2\pi r^2}\left\{\frac{e^{-kr}}{r^3}(1 + kr) - \frac{\sigma_n}{\sigma_t}\frac{e^{-kr'}}{(r')^2}\left(k + \frac{1}{r'}\right)\right\}$$

$$+ \frac{Il}{2\pi r^4}\left\{xy\left[\frac{kr}{2}\left(I_0K_1 - I_1K_0\right) - I_1K_1\right]\right\} \qquad (14)$$

$$H_y = -\frac{Il}{2\pi r^4}\left[(3y^2 - x^2)I_1K_1 + \frac{y^2}{2}(kr)\left(I_1K_0 - I_0K_1\right)\right] \qquad (15)$$

$$H_z = \frac{3Ily}{2\pi k^2 r^5}\left[1 - e^{-kr}\left(1 + kr + \frac{k^2r^2}{3}\right)\right] \qquad (16)$$

In the preceding equations, the functions I_0, K_0, I_1 and K_1 are the modified Bessel functions of the first and second kind and the first order. In these equations for $z = 0$, the argument of the modified Bessel functions is $kr/2$.

From these equations for infinitesimal dipoles, computer solutions were obtained for dipoles of finite length by the superposition of the opposite charges of the infinitesimal dipoles in the method that was mentioned in the introduction to this paper. For a finite length of dipole, the values of the fields for H_x and E_x are presented by the curves in Figures 2 and 3, respectivelv. The curves are for a finite dipole of 100 m length and of unit moment.

Results and Conclusions

The exact equations for an infinitesimal dipole that are given in the text were combined by use of a computer, as described in the introduction, to obtain the fields for dipoles with finite length and unit moment. The accompanying curves present the results for two fields, H_x and E_x, which are two of the five fields for which equations are given in the text. Since the accompanying results are for only one of the almost 12 solutions for different applications of the field and for different earth characteris-

In equation (12), the number in the first parentheses should be 2 instead of 3.

HORIZONTAL ELECTRIC DIPOLE OF 100 METERS LENGTH ON ANISOTROPIC EARTH

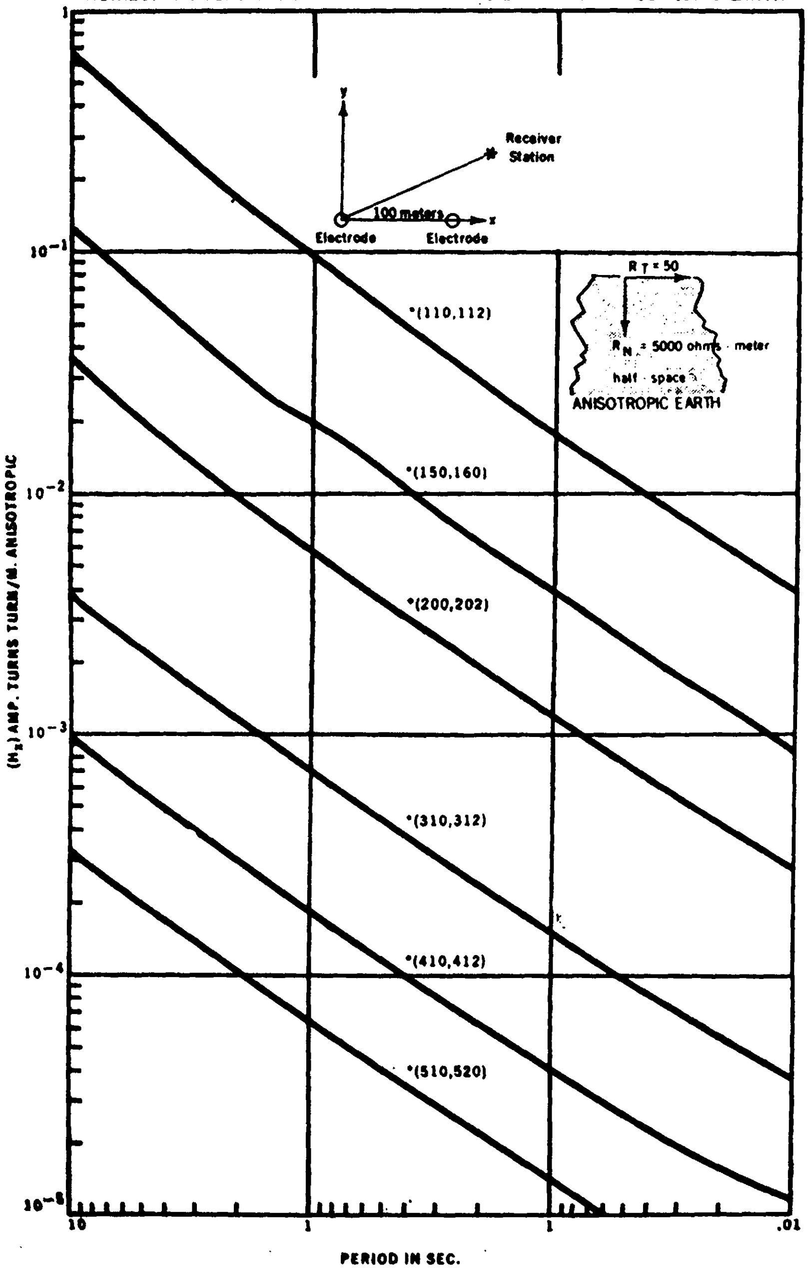

Figure 2

COMPARISON OF THE E_x FIELD FOR ISOTROPIC AND ANISOTROPIC EARTH

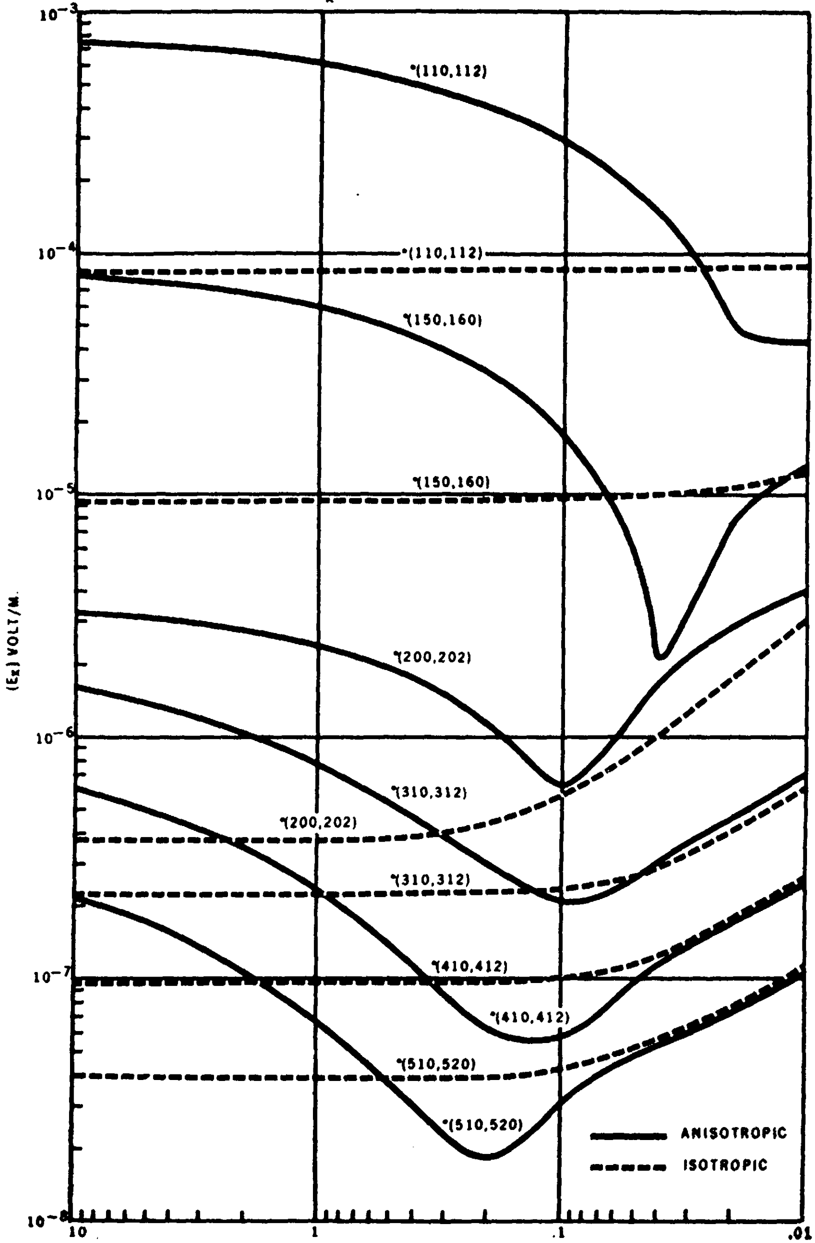

Figure 3

tics, these results are to be interpreted only in broad terms. Detailed differences may be clarified by the other solutions.

The qualitative variation of the magnetic component of the field, H_x, parallel to the axis of the dipole is not unusual. The intensity of the field at various distances from the dipole is shown in Figure 2 as a function of the distance and the logarithm of the frequency. The frequency range is 0.1 to 100 cycles/sec. This component of the magnetic field decreases exponentially with the distance and with the frequency, as would be expected. The decrease of the field with frequency corresponds with a decrease of the total current outside of the symmetrical region that is defined by the position of the receiver.

The quantitative variation of the electric gradient in the direction of the axis of the dipole is compared for a finite dipole in an isotropic earth (5000 ohm-meters in every direction) and a finite dipole in an anisotropic earth (50 ohm-meters transverse and 5000 ohm-meters normal to the surface). Two factors produce the major effect in the results, the anisotropic resistivity and the interaction of the magnetic fields with the current. This interaction is familiar under different conditions as the skin effect and the pinch effect. The fields decay with distance from the dipole as expected. For the isotropic earth, the field is almost constant at the lower frequencies. Although the total current outside of the receiver is decreasing, the remaining current is forced inward toward the axis of the dipole and upward to the surface of the earth as the frequency increases. The rise in the gradient at higher frequencies is a short ascendency of the skin effect in forcing the current to the surface and the signal will again decrease at still higher frequencies. The dip in the field for the anisotropic earth is a consequence of the smaller fraction of the current that penetrates the earth.

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