The Group of Symmetries of a Regular Tetrahedron

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INTRODUCTION

The purpose of this investigation was to learn how the system of the symmetries of a regular tethahedron conforms to a known mathematical structure, the group. Through the development of elements and a binary operation, the properties of the system are elucidated.

The study of a particular example of a group, not explained in books on the subject, gives a different view of the group itself. In this report, all presented aspects are given complete coverage but, because of my varied interests, some concepts were studied in a more intense light than is customary. An example of this is the development of elements.

McCoy (1964) and Dubish (1965) gave thorough explanations of the group in general. However, none of the books read discusses the movements of a three-dimensional object forming a group. In part, the absence of this information led me to this investigation.

EXPERIMENTAL PROCEDURE

In examining the symmetries of a regular tetrahedron, a physical model (Fig. 1) was useful.

A regular tetrahedron is a four-sided polygon having equilateral triangles as faces. There are four axes, each passing through a vertex and each perpendicular to the face opposite the vertex.

In order to perform operations on the tetrahedron, it is necessary to be able to identify each vertex on the model by labeling as numbers *1*, *2*, *3* and *4*. It is to be noted that a vertex of the tetrahedron is made up of the intersection of three planes. Therefore, three faces have a vertex labeled *1*, for example. With this information, the four axes of the regular tetrahedron are defined as follows:

AXIS ONE — the line passing through vertex *1* and perpendicular to the face with vertices *2*, *3* and *4*.

AXIS TWO — the line passing through vertex *2* and perpendicular to the face with vertices *3*, *1* and *4*.

AXIS THREE — the line passing through vertex *3* and perpendicular to the face with vertices *2*, *1* and *4*.

AXIS FOUR — the line passing through vertex *4* and perpendicular to the face with vertices *3*, *1* and *2*.

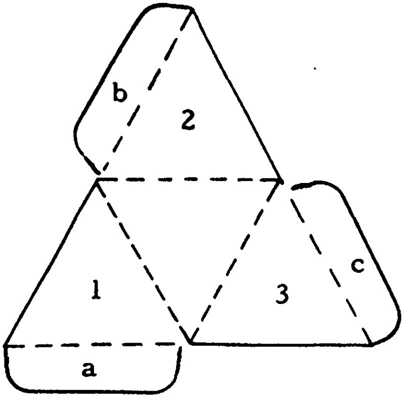

Fig. 1. Model tetrahedron. Sections 1, 2 and 3 are folded up to form a peak, then flaps *a, b* and *c* are glued inside the tetrahedron.

The elements of the intended group may now be explained. Each element represents one or more rotations about one or more axes. Therefore, an element may be defined as a particular motion. The result of the motion is always found on the bottom face which is designated by a series of four numbers. These numbers signify the vertices of the face *as seen from underneath the tetrahedron* (Fig. 2), and the vertex which is not of the bottom face. For example, (*3 1 2 4*) means the bottom face looks like Fig. 2, with vertex *4* on the other side of the tetrahedron.

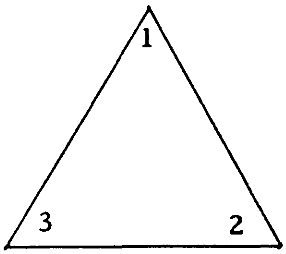

Fig. 2. The numbered vertices as seen from underneath the tetrahedron.

The first digit in the connotation (*3 1 2 4*) is the number of the lower left-hand vertex. The second digit is the number of the center vertex, and the third digit is the number of the lower right-hand vertex. The fourth figure is the number of the remaining vertex. This method of designating faces is applicable to any face of the tetrahedron, regardless of whether it is the bottom one.

A motion may be defined by describing the actual physical process or by using permutation symbols. By the first method,

> a = With the tetrahedron lying on the original face (*1 2 3 4*), it is rotated counter-clockwise (looking from vertex *1* for 120° about axis *1*.

If the tetrahedron is carefully picked up so that its vertices are not rearranged, the result of the motion may be found by looking at what was its bottom face before it was picked up, i.e. (*1 3 4 2*).

The second method, that is, using permutation symbols, is defined as follows:

$$\alpha = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{vmatrix}$$

The first line of digits in this symbol gives the position of the four vertices of the tetrahedron, the first three being those of the original bottom face. The second line does the same thing for the final bottom face after the motion is carried out, i.e. (*1 3 4 2*). In other words, under the mapping α, the image of *1* is *1*, the image of *2* is *3*, the image of *3* is *4*, and the image of *4* is *2*.

By these two methods, the remaining eleven elements are defined (Table I).

An operation performed on any two of these 12 elements is denoted by an asterisk (*) and is defined as performing one motion and then the other, as specified. This may be better understood if expressed as an equation:

$$\overset{\alpha}{\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{vmatrix}} * \overset{\beta}{\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{vmatrix}} = \overset{\gamma'}{\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{vmatrix}}$$

When the actual model of the tetrahedron with the labeled vertices is used, Face (*1 2 3 4*) is placed on the bottom. Then α is performed, i.e., the tetrahedron is rotated counter-clockwise (looking from vertex *1*) for 120° about axis *1*. This brings (*1 3 4 2*) to the bottom face. Next, β is performed with disregard to the condition of the bottom face. The tetrahedron is rotated counter-clockwise (looking from vertex *2*) for 120° about axis *2*. Now the bottom face is examined. The result of an operation is the element having the same result as the bottom face of the tetrahedron after the operation (the two motions) has been performed. In this case, the result is element γ', since (*4 1 3 2*) is the bottom face after the motion γ' has been executed on the original face which is *always* (*1 2 3 4*).

However interesting this method is, the method of performing operations using permutation symbols is simpler and more effective. It is scarcely practical to perform 144 actual physical operations when the results may be obtained in one-tenth the time. Using the example, one column of the elements may be taken at a time as follows:

$$\overset{\alpha}{\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{vmatrix}} * \overset{\beta}{\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{vmatrix}}$$

α is looked at first. In this element, the following substitutions occur:

1—>*1*, *2*—>*3*, *3*—>*4*, and *4*—>*2*. In β, the substitutions are: *1*—>*4*, *2*—>*2*, *3*—>*1*, and *4*—>*3*. Combining α and β, the following is obtained:

1—>*1*—>*4*, *2*—>*3*—>*1*, *3*—>*4*—>*3*, and *4*—>*2*—>*2* *or*
1 ——— >*4*, *2* ——— >*1*, *3* ———>*3*, and *4* ——— >*2*.

Therefore the motion which is the result of the operation may be expressed by:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{vmatrix}$$

This is precisely what element γ' is. Hence, γ' is the result of the operation * on α and β.

For a complete table of the operations and their results, refer to Table II.

TABLE I. ELEMENTS

Note: The original bottom face is (*1 2 3 4*).

ELEMENT	PHYSICAL DESCRIPTION	SYMBOL
ε	The tetrahedron is rotated counter-clockwise (looking from the corresponding vertex) for 360° about axes *1, 2, 3* and *4*. Result: (*1 2 3 4*)	$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{vmatrix}$
α	The tetrahedron is rotated counter-clockwise (looking from vertex *1*) for 120° about axis *1*. Result: (*1 3 4 2*)	$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{vmatrix}$
α'	The tetrahedron is rotated counter-clockwise (looking from vertex *1*) for 240° about axis *1*. Result: (*1 4 2 3*)	$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{vmatrix}$
β	The tetrahedron is rotated counter-clockwise (looking from vertex *2*) for 120° about axis *2*. Result: (*4 2 1 3*)	$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{vmatrix}$
β'	The tetrahedron is rotated counter-clockwise (looking from vertex *2*) for 240° about axis *2*. Result: (*3 2 4 1*)	$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{vmatrix}$
γ	The tetrahedron is rotated counter-clockwise (looking from vertex *3*) for 120° about axis *3*. Result: (*2 4 3 1*)	$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{vmatrix}$
γ'	The tetrahedron is rotated counter-clockwise (looking from vertex *3*) for 240° about axis *3*. Result: (*4 1 3 2*)	$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{vmatrix}$
δ	The tetrahedron is rotated counter-clockwise (looking from vertex *4*)for 120° about axis *4*. Result: (*3 1 2 4*)	$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{vmatrix}$
δ'	The tetrahedron is rotated counter-clockwise (looking from vertex *4*) for 240° about axis *4*. Result: (*2 3 1 4*)	$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{vmatrix}$
λ	The tetrahedron is rotated counter-clockwise (looking from vertex *3*) for 240° about axis *3*. Then it is rotated about axis *2* counter-clockwise (looking from vertex *2*)for 120°. Result: (*3 4 1 2*)	$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{vmatrix}$
μ	The tetrahedron is rotated counter-clockwise (looking from vertex *1*) for 240° about axis *1*. Then it is rotated about axis *3* counter-clockwise (looking from vertex *3*) for 120°. Result: (*2 1 4 3*)	$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{vmatrix}$
ν	The tetrahedron is rotated counter-clockwise (looking from vertex *2*) for 240° about axis *2*. Then it is rotated about axis *1* counter-clockwise (looking from vertex *1*) for 120°. Result: (*4 3 2 1*)	$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{vmatrix}$

TABLE II. OPERATIONS

Note: An element in the left-most column is performed first with an element in the top row performed second.

	ϵ	α	α'	β	β'	γ	γ'	δ	δ'	λ	μ	η
ϵ	ϵ	α	α'	β	β'	γ	γ'	δ	δ'	λ	μ	η
α	α	α'	ϵ	γ'	λ	δ'	η	β'	μ	δ	γ	β
α	α'	ϵ	α	η	δ	μ	β	λ	γ	β'	δ'	γ'
β	β	δ'	λ	β'	ϵ	α'	μ	γ'	η	γ	δ	α
β'	β'	η	γ	β	β	λ	δ	μ	α	α'	γ'	δ'
γ	γ	β'	η	δ'	μ	γ'	ϵ	α'	λ	β	α	δ
γ'	γ'	μ	δ	λ	α	ϵ	γ	η	β	δ'	β'	α'
δ	δ	γ'	μ	α'	η	β'	λ	δ'	ϵ	α	β	γ
δ'	δ'	λ	β	μ	γ	η	α	ϵ	δ	γ'	α'	β'
λ	λ	β	δ'	α	γ'	δ	β'	γ	α'	ϵ	η	μ
μ	μ	δ	γ'	γ	δ'	β	α'	α	β'	δ	ϵ	λ
η	η	γ	β'	δ	α'	α	δ'	β	γ'	μ	λ	ϵ

DISCUSSION

The 12 elements may be grouped together in a set G with operation * and an equivalence relation $=$. As there are only 12 possible ways for the tetrahedron to be positioned, and 12 elements representing these ways, the property of *closure* holds because the result of the operation on any two elements is an element of G. In other words, if one motion is followed by another motion, the resulting position could have been obtained as a single motion.

The 12 given elements are substitutions of the vertex numbers of the tetrahedron. Therefore, if it is proved that all substitutions are associative, then this system of the symmetries of a regular tetrahedron will be associative.

PROOF:

Given $a, b, c \in G$. TO PROVE: $a * (b * c) = (a * b) * c$

$$a = \begin{vmatrix} - & - & M & - \\ - & - & N & - \end{vmatrix}$$

$$b = \begin{vmatrix} - & - & N & - \\ - & - & O & - \end{vmatrix}$$

$$c = \begin{vmatrix} - & - & O & - \\ - & - & P & - \end{vmatrix}$$

$$a * b = \begin{vmatrix} - & - & M \\ - & - & O \end{vmatrix} \qquad b * c = \begin{vmatrix} - & - & N & - \\ - & - & P & - \end{vmatrix}$$

$$\text{Therefore } [\ a * (b * c)] = \begin{vmatrix} - & - & M & - \\ - & - & N & - \end{vmatrix} * \begin{vmatrix} - & - & N & - \\ - & - & P & - \end{vmatrix} = \begin{vmatrix} - & - & M & - \\ - & - & P & - \end{vmatrix}$$

$$[(a * b) * c)] = \begin{vmatrix} - & - & M & - \\ - & - & O & - \end{vmatrix} * \begin{vmatrix} - & - & O & - \\ - & - & P & - \end{vmatrix} = \begin{vmatrix} - & - & M & - \\ - & - & P & - \end{vmatrix}$$

$$\begin{vmatrix} - & - & M & - \\ - & - & P & - \end{vmatrix} = \begin{vmatrix} - & - & M & - \\ - & - & P & - \end{vmatrix}$$

All substitutions are associative. Therefore, this system is *associative.*

Element ε illustrates the *existence of an identity.* If ε is the element combined under the operation * with any other element, the result is always the "other element". This is because the element ε does not change the position of the tetrahedron.

In the Cayley array which comprises Table II, the identity element ε appears once and only once in each column and in each row. Since the appearance of an identity element indicates that two elements, in effect, have cancelled each other out, and are therefore the inverses of each other, *there exists an inverse for each element of G.*

Any mathematical system for which these four properties hold is termed a group. Therefore, the system of the symmetries of a regular tetrahedron may be classified as a *group.*

Conclusion

This investigation has expanded group theory into the realm of three-dimensional figures instead of the conventional planar figures. In so doing, certain concepts have been elucidated, for example, the development of elements and an operation.

A promising expansion of this system would be the development of elements on the symmetries of an n-gon, thereby changing from the finite to the infinite.

References

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