An Algebraic Ring of F-Sequences and Determinant Solutions to Simultaneous Equations in the System

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PREFACE

The purpose of the project was to build a mathematical system around the Fibonacci sequence. The elements that resulted are called *F*-sequences and are denoted (x,y) where x and y are real numbers and are the first two terms of a sequence. These elements are used to prove many interesting properties which show that the system is an algebraic ring. The resulting capabilities of such a system were proved, in that simultaneous equations could be set up and their solutions found on the IBM 1620 computer. Sets of three equations had *F*-sequence coefficients and constants and the variables were *F*-sequences.

F-SEQUENCES

There is an obscure set of numbers known by mathematicians as the "Fibonacci Sequence". It has many rare properties and has recently been found to have application to many natural phenomena. The first two terms are 0 and 1, and every term thereafter is the sum of the preceding two terms. Thus, the rule for forming this requence is: $f_{n+2} = f_{n+1} + f_n$, where $f_0 = 0$, and $f_1 = 1$.

The only difference between the Fibonacci sequence and any other formed in a similar manner is the set of two beginning terms. The universe, U, of *F*-sequences is the set of all sequences for which the rule for forming is: $u_{n+2} = u_{n+1} + u_n$, and for which $u_0 = p$, $u_1 = q$, where p and q are any real numbers. Since the first two terms determine the sequence, then an *F*-sequence will be denoted as follows: (p,q).

One of the unusual properties of an *F*-sequence is that the limit of the ratios of consecutive terms is equal to the famous "golden ratio". That is $\lim (u_{n+1}/u_n) = (1 + 5^{1/2})/2$.

If A represents the sequence $(a_0,a_1,\ldots)$, B represents the sequence $(b_0,b_1,\ldots)$ and $A + B$ is defined to be the sequence $(a_0 + b_0, a_1 + b_1,\ldots)$, xa is defined to be the sequence $(xa_0,xa_1,\ldots)$. It is easy to show that the set of sequences with these two operations form a vector space. If $a_{n+2} = a_{n+1} + a_n$, $b_{n+2} = b_{n+1} + b_n$, $n = 0, 1,\ldots$, and x and y are real numbers, then $xa_{n+2} + yb_{n+2} = xa_{n+1} + yb_{n+1} + xa_n + yb_n$, so the *F*-sequences also form a vector space. Moreover, the first two terms are $xa_0 + yb_0$ and $xa_1 + yb_1$. That is, in the notation above, $x(p,q) + y(u,v) = (xp + yu, xq + yv)$, and the vector space of *F*-sequences is the same as the ordinary vector space of pairs of real numbers.

Henceforth, F, G, H, will denote the particular *F*-sequences (0,1), (1,0), and $(1,1) = F + G$. Any *F*-sequence $U = (p,q)$ can be written as $pG + qF$ or $(q-p)F + pH$. If U is an *F*-sequence, U_n will denote the *F*-sequence which begins with the term u_n of U. That is, if $U = (p,q)$, $U_0 = U$, $U_1 = (q,p+q)$, $U_2 = (p+q,p+2q)$,... Then $U = (q-p)F + pH$ can be written $U = (q-p)F_0 + pF_1$. This suggests that a product might be defined for *F*-sequences, so that F would be an identity, by $UV = (q,p)\ V_0 + pV_1$. If $V = (r,s)$ then $UV = (q-p)\ (r,s) + p(s,r+s)$, or $(p,q)\ (r,s) = (qr + ps - pr, pr + qs)$. Straightforward verification of the commutative, associative, and distributive laws shows that with these definitions of sum and products, the F-sequences form a commutative ring with F as multiplicative identity.

Moreover, for any F-sequence $U = (p,q)$ multiplication by powers of H has the same effect as cancelling the first terms of a sequence according to the power. Specifically, if u_n denotes the nth term of the sequence, then $H^nU = U_n$ or $(1,1)^n(p,q) = (1,1)^n(u_0,u_1) = (u_n,u_{n+1})$. This can be proved by induction since $(1,1)\ (p,q) = (q,p+q) = U_1$ and if $H^kU = U_k$ then $H^{k+1}U = H(H^kU) = HU_k = (1,1)\ (u_k,u_{k+1}) = (u_{k+1},u_{k+1} + u_k) = (u_{k+1},u_{k+2}) = U_{k+1}$.

Some interesting properties of F-sequences can be derived by the use of the ring properties. It is customary to define the zero exponent to be the identity so for any non-zero F-sequence U, U^0 will be defined as F. Since $G\ H = F$, H^{-n} can be defined as G^n. The first few powers of G, namely $G^0 = (0,1)$, $G^1 = (1,0)$, $G^2 = (-1,1)$, $G^3 = (2,-1)$, $G^4 = (-3,2)$, $G^5 = (5,-3)$, indicates that $G^n = [(-1)^{n+1}f_n,(-1)^nf_{n-1}]$. This can be shown by induction because if it is true for n, then $G(G^n) = (1,0)\ [(-1)^{n+1}f_n,(-1)^n(f_{n-1})] = [(-1)^{n+2}f_{n+1}(-1)^{n+1}f_n]$ and it is true for $n+1$. It is true for $n = 0$, so for all $n = 0,1,2,3,\ldots$. Two well-known identities about F-sequences follow quickly from algebraic identities in the ring. Since $F_mF_n = H^mFH^nF = H^{m+n}F = F_{m+n} = (f_m,f_{m+1})\ (f_n,f_{n+1}) = (f_mf_{n+1} + f_{m+1}f_n - f_mf_n,f_mf_n + f_{m+1} + f_{n+1}) = (f_{m+n},f_{m+n+1})$, it follows from equating the first terms in the last two expressions and using the fact that $f_{n+1} - f_n = f_{n-1}$ that

$$f_{m+n} = f_mf_{n-1} + f_{m+1}f_n.$$

Moreover, since $H^nG^n = F$, or $(f_n,f_{n+1})[(-1)^{n+1}f_n,(-1)^nf_{n-1}] = [(-1)^n(f_nf_{n-1} - f_{n+1}f_n + f_n^2),\ (-1)^{n+1}(f_n^2 - f_{n-1})] = [0,\ (-1)^{n+1}(f_n^2 - f_{n+1}f_{n-1})] = (0,1)$ it follows that

$$f_n^2 - f_{n+1}f_{n-1} = (-1)^{n+1}.$$

In order to see what was involved in solving systems of equations in this ring, a system was programmed for the IBM 1620 computer to solve a system of three equations in three unknowns with coefficients and constants for F-sequences, by using Cramer's rule where the solution for an unknown is the quotient in which the determinant obtained by replacing the column of coefficients of the unknown by the column of constants is divided by the determinant of the coefficients. It was found that it was first necessary to know how to divide F-sequence or how to find the reciprocal of an F-sequence. This, however, reduces to solving two equations in two unknowns with real coefficients. That is, if $U^{-1} = (x,y)$ is sought where $U = (p,q)$ then

$$(q - p)x + py = 0,\ px + qy = 1.$$

The solution can be found if $q^2 - p^2 - qp$ is non-zero, or if q is not $[(1 \pm 5^{1/2})/2]p$.