Multistructured Mathematical Models in Water Resources Analysis and Design

GEORGE W. REID, University of Oklahoma, Norman

The engineer is called upon to design either a single entity or a complex grouping of entities that propose to provide certain useful services at some future date. The particular systems design involving usually natural phenomena has never been undertaken before and is too large to be experimented with. Generally, the physical principles and natural phenomena are not altogether understood, and their mathematical equations are both complex and of dubious value. To circumvent these major "road blocks", detailed studies of previously designed and operated systems are studied and judicious transposition is undertaken. Realizing the inadequacy of this procedure, engineers have resorted to physical or iconic models of the proposed systems. In structural problems true models, with the entire proposed structures built at a reduced scale, have been used. In hydraulic systems, distorted or adequate models, employing several scales at once, have been developed to provide prediction equations. Most of the major projects of the Corps of Engineers have been modeled in this fashion at Vicksburg and operated under varying conditions. In Germany in the 1930's, I noted that each major river system had an iconic model, kept up to date. Because of the limitations of this approach, analogs were used on some water resources problems such as distribution systems analysis and ground water draw-down recovery studies. Herein, an analog between head loss (h), flow (Q), and resistance (r) on one hand and voltage (E), current (I) and resistance (R) on the other was used to predict system performance.

The mathematical relationships between h, Q, and r were not analogous to those of E, I, and R unless a nonlinear system was used or adaptive equations were developed. Both techniques were used and continue to be used, though high speed digital computers are equally applicable to these problems wherein the "Hardy-Cross" system of successive approximation is solved by machine iteration.

The system concept is a very general one and underlies the various techniques for simulating behavior. A variation in the input will bring about a particular variation or response in the output. The dependence of output on input is defined by the relationships within the system. In water resource development problems the total input-output relationships involve several variants and stages, multivariant-multistaged. The complexity is the result of the arrangement of the individoal elements, rather than their intrinsic nature. Thus, it is not the complicated nature of the individual events that causes complexity, but rather the combination of a large number of effects, both simple and minor.

A water resource system, consisting of benefits to be gained from power, irrigation, flood control, low-flow augmentation, and costs for impoundment and treatment, presents a typical problem. At present, many of these studies are conducted by combining on a one-to-one basis the responses to the stimuli essentially under conditions of isolation. The technique involves the ordering or ranking of several alternate combinations of reservoirs and uses to arrive at the best alternate. If individual performance of several reservoir costs to provide yields are known, combination yields are in turn appraised in terms of benefits from combinations of uses, power, irrigation, etc. The reactions in a given part, unfortunately, depend not only upon what is going on in it, but also upon the state of the whole system, or in reality a system of dynamically interacting elements within a complex web of ecological relationships. Every real system has an infinitely large number of possible inputs and outputs. The designer is seeking that unique one optimal set of inputs and outputs. The behavior of a system may be studied by means of an isomorphic one, analogous in pattern, such as a mathematical model. A mathematical model is simply a symbolic representation of a physical system or a set of linear or nonlinear relationships between variables, measurable or not subject to inequalities or constraints. A multistructured model depicts a system involving a sequence of decisions or responses over time or space, or a combination of both. The information pattern devotes the accumulation of knowledge about the system as a consequence of responses to decisions. Thus there is a feedback process. Dynamic programming is a mathematical formulation of the feedback process that arises naturally in the multistage decision process, a multistaged decision process and a sequential optimization procedure. A policy is the procedure used to determine responses. Simply, the policy tells one what to do next in terms of where one is and what is known.

Generally, models are divided into two categories, deterministic and probabilistic. In the deterministic model the adoption of a certain policy is known to lead to, or assumed to lead to, a specific outcome or value of the objective function. The model is built, or the policy established, with sets of decision variables, or noncontrollable variables (exogenous), and parameters that are completely determined. Herein, the chance occurrence of variables is ignored and the model is considered to follow a definite law of certainty. On the other hand, the probabilistic model does not meet these requirements, but usually is one based on accumulated statistical data dictating the policy. The process is time-independent, and the sequence of occurrence is ignored. Conventional flood routing or the unit hydrograph is deterministic while the flow-duration-curve procedure is probabilistic.

Models can take many forms, running the gamut from differential equations subject to analytical solutions, to sets of nonlinear equations, requiring either linear or dynamic programming techniques and simulation analyses. Solution techniques are usually divided into graphical, algebraic, and/or iterative methods. Graphical solutions are usually limited to problems involving one or two independent variables. Algebraic solutions include nearly all integral and differential equations, as well as linear programming. In simulation analyses the model builder is no longer restricted to models for which he devises analytical solutions. With the use of high-speed machines and special iterative techniques, all possible solutions of the various inputs can be studied and optimal results obtained.

Setting up and solving a mathematical model should be standard operating procedure before attempting a design. Many situations can now be examined and explored and, despite the frequent necessity of skeletonization, valuable screening can be accomplished if not an optimal solution.

Several examples might better illustrate the construction of Water Resource Models. Consider a simple inventory model wherein water need be stored for a seasonal irrigation and used periodically: the storage space has a cost per unit/yr. (w) associated with it; the storage facility investment also has interest and associated costs (i); and the periodic use has use costs (a). Now let (d) be the annual demand for water, (q) the periodically required quantity, and finally (c) the purchase price per acre-ft. The mathematical model where all parameters are determined is:

$$\text{Total annual cost } (\$) = a(d/q) + cd + (icq/2) + wq \qquad (1)$$

To find the value of (q) to minimize $(\$)$ the equation is differentiated with respect to (q) and set equal to zero; then solved for (q).

$$d(\$)/dq = -\ (ad/q^2Q) + 0 + (ic/2) + w \qquad (2)$$

and

$$q = (2ad/ic + 2w)^{1/2} \qquad (3)$$

More explicitly, if $d = 2000\ AF$, $c = \$1.00/AF$,

$i = 0.16$, $a = \$10.00$, and finally $w = \$0.10$

then $q = 333\ AF$

Similarly, a brine pollution gradient can be modeled in differential equation form

$$dc/dt = (Ed^2C/dx) - (Udc/dx) - kc \qquad (4)$$

wherein the gradient equals diffusion less evection and reaction. The equation can be evaluated when $(dc/dt) = 0$, at steady state, providing a useful dimensionless constant

$$KE/U^{n} \qquad (5)$$

As an example of a model requiring a linear programming assume that water can be used for irrigation and/or for power, at a benefit of \$2/*AF* and \$8/*AF* respectively. Assume further that of the total 500 *AF* available, only 400 *AF* can be used for irrigation, and only 300 *AF* can be used for power.

Maximize the Benefit Function:

$$\text{Max } f(x_1 x_2) = 2x_1 + 5x_2 \qquad (6)$$

Subject to the following constraints:

x_1 equal to or less than 400
x_2 equal to or less than 300
$x_1 + x_2$ equal to or less than 500
x_1, x_2 equal to or greater than zero.

The graphical solution is shown in Fig. 1.

The algebraic and simplex solution are based on the completion of the inequalities with slack variables as follows:

$$Z = 2x_1 + 5x_2 + 0(s_1 + s_2 + s_3) \qquad (7)$$
$$x_1 + s_1 = 400$$
$$x_2 + s_2 = 300$$
$$x_1 + x_2 + s_3 = 500$$

and a rather simple stepwise procedure is used to optimize z.

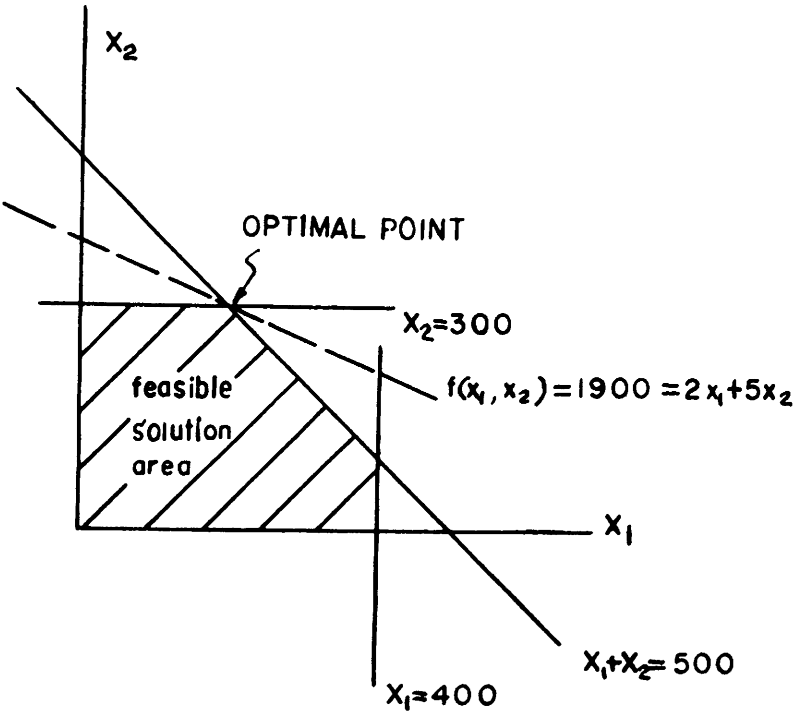

FIG. 1

One highly interesting model is the one developed to optimize waste treatment and low flow augmentation wherein the input-output relationships are nonlinear and require simulation analyses. They are shown schematically in Fig. 2.

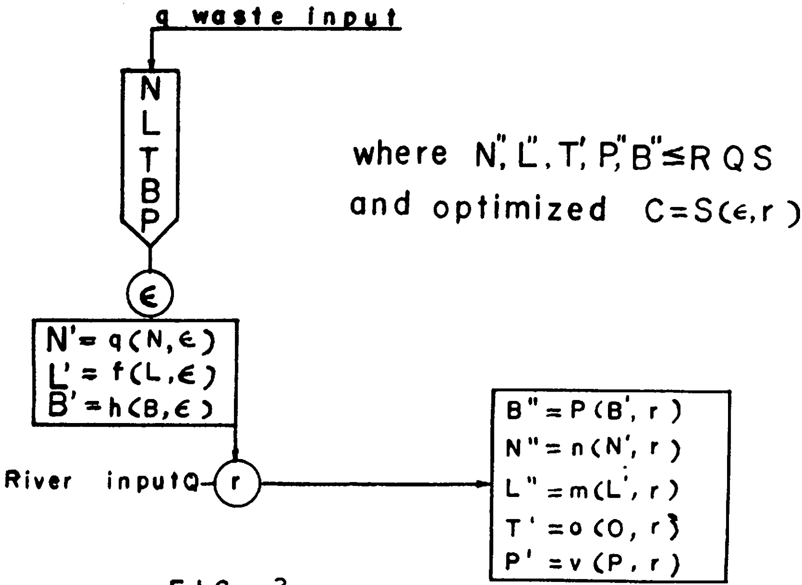

FIG. 2

N, L, T, B, and *P* represent nutritional biodegradable, thermal, bacterial and persistent chemical pollution. The *r* represents the dilution ratio of stream flow to waste flow necessary for certain stream characteristics to provide an acceptable water quality (*RQS*) and ϵ represents the efficiency of the treatment.

The previous model is not constructed to include the statistical variability of the river input which would require the generation of synthetic hydrographs. The generation of a synthetic hydrograph involves the use of a stochastic model. Actually such a model incorporates both deterministic and probabilistic elements. A typical form:

$$Q_{n+1} = \bar{Q} + \alpha\,[Q_i - \bar{Q}] + \beta\Sigma_e\,[1-r^2]^{1/2} \qquad (8)$$

wherein $\bar{Q} + \alpha\,[Q_i - \bar{Q}]$ assures a linear relationship between successive periods, the deterministic elements, and $\beta\Sigma_e\,(1-r^2)^{1/2}$ estimates the unexplained variance, the random component.

New techniques, dating in some respects to simulation processes called "War Games", decision theory, and linear and dynamic programming, made possible by high speed machines, have made what has been taken for granted as a world with a geometry of three dimensions, now one with a geometry of as many dimensions as variables, each being equivalent to a coordinate. Single and multiple systems models may be built and analyzed for every possible policy.