Application of Matrix Theory to Some Problems in Oscillatory Motion

R. D. HARTMAN[1] and Y. S. KIM[2]

Department of Physics, Oklahoma State University, Stillwater

Matrix theory, or more generally linear algebra, is a relatively recent mathematical development. Its roots extend back 100 years to the work of Hamilton, Cayley and Sylvester (Finkbeiner, 1960) but it has attracted widespread interest only in the past two or three decades. The growth of its applications and usefulness has been remarkable. Although other branches of higher mathematics may be applied more intensively, and perhaps by more people, few branches have been applied to so many diversified fields as has the theory of matrices. Today, matrices are effective tools in such disciplines as psychology, education, chemistry, physics, engineering, mathematics, and economics, just to name a few. In particular, matrices are widely used in microscopic processes, such as encountered in nuclear interactions, wave mechanics (Schiff, 1955) and molecular spectroscopy (Wilson, et al., 1955).

In a series of papers (Adem and Moshinsky, 1952; Moshinsky, 1951; Adem, 1959) several macroscopic physical processes are described by column vectors or matrices instead of single functions; thus, a simplification in the mathematical solution of such processes has been achieved and, hence, has made possible the solution of a variety of problems with matrix techniques.

The purpose of this paper is to apply matrix theory to some problems involving macroscopic physical systems which are undergoing one-dimensional oscillation. The application of matrices consists of the following: (1) form a real symmetric matrix, A, which shall be called the configuration matrix, from the equations of motion of the system in question; (2) solve the eigenvalue equation equation; (3) determine the eigenvectors; (4) normalize the eigenvectors; and (5) perform an orthogonal transformation on the configuration matrix. The general solution of the differential equation of motion is the sum of the particular and the complementary solutions as usual.

[1]National Science Foundation Science Faculty Fellow 1965-66.
[2]Present address: Soong Sill Academy, Yongsan, Seoul, Korea.

ONE-DIMENSIONAL UNIFORMLY ACCELERATED SYSTEM

Consider a system of coupled oscillators consisting of three masses m_1, m_2 and m_3 connected in series by four weightless, noninteracting springs of force constants k_1, k_2, k_3, and k_4 respectively. Let the loose ends of springs 1 and 4 be rigidly fastened to a weightless frame. Mount this linear system of coupled oscillators horizontally in the X-Y plane, and impart a uniform acceleration, a, to the system by tranporting the weightless frame along the $+X$ axis.

The equations of motion for the mass system are[3]:

$$\begin{aligned} m_1x''_1 &= -k_1x_1 + k_2(x_2-x_1) + m_1a \\ m_2x''_2 &= k_2(x_1-x_2) + k_3(x_3-x_2) + m_2a \\ m_3x''_3 &= -k_3(x_3-x_2) - k_4x_3 + m_3a \end{aligned} \tag{1}$$

where x''_i represents the second time derivative of x_i ; and x_i represents the displacement of mass m_i from equilibrium.

For simplicity let $m_1 = m_2 = m_3$, and $k_1 = k_2 = k_3 = k_4 = k$. Further let $b = k/m$. Then Eq. (1) becomes

$$\begin{aligned} x''_1 &= -2bx_1 + bx_2 \qquad\qquad + a \\ x''_2 &= bx_1 - 2bx_2 + bx_3 + a \\ x''_3 &= \qquad\quad bx_2 - 2bx_3 + a \end{aligned} \tag{2}$$

Using matrix notation, Eq. (2) may be written as

$$X'' = AX + G \tag{3}$$

where:

$$X = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} ; \; A = \begin{vmatrix} -2b & b & 0 \\ b & -2b & b \\ 0 & b & -2b \end{vmatrix} ; \text{ and } G = \begin{vmatrix} a \\ a \\ a \end{vmatrix} \tag{3a}$$

Thus, step 1 of the procedure, namely that of forming a real symmetric matrix from the equations of motion of the system, has been accomplished. Next, we proceed to the eigenvalue equation.

Consider the complementary solution of Eq. (3), as found by setting $G = 0$. Then

$$X'' = AX \tag{3b}$$

The eigenvalue equation for A is given by:

$$AY_i = \lambda_i Y_i \tag{4}$$

where Y_i is the eigenvector and λ_i is the eigenvalue. The characteristic equation for Eq. (4) is:

$$(A - \lambda I)Y_i = 0 \tag{5}$$

where I represents the identity matrix. In order for Eq. (5) to hold for all Y_i we require the determinant of the coefficient of Y_i to be zero. Hence, from Eqs. (3a) and (5) we obtain:

$$\det \begin{vmatrix} -2b-\lambda & b & 0 \\ b & -2b-\lambda & b \\ 0 & b & -2b-\lambda \end{vmatrix} \tag{6}$$

[3]Here, the Lagrangian technique for obtaining the equations of motion has been employed. The last term in each of the equations is the contribution of the linear acceleration of each mass to the potential energy term.

Solving Eq. (6) gives for the eigenvalues:

$$\begin{aligned} \lambda_1 &= -2b \\ \lambda_2 &= [-2 + (2)^{1/2}]b \qquad (7) \\ \lambda_3 &= [-2 - (2)^{1/2}]b \end{aligned}$$

The corresponding eigenvectors as obtained from Eq. (4) are:

$$(8) \qquad Y_1 = \begin{vmatrix} (\frac{1}{2})^{1/2} \\ 0 \\ -(\frac{1}{2})^{1/2} \end{vmatrix} ; \; Y_2 = \begin{vmatrix} \frac{1}{2} \\ (\frac{1}{2})^{1/2} \\ \frac{1}{2} \end{vmatrix} ; \text{ and } Y_3 = \begin{vmatrix} \frac{1}{2} \\ -(\frac{1}{2})^{1/2} \\ \frac{1}{2} \end{vmatrix}$$

Thus steps 2 and 3 have been achieved.

It may be verified that the eigenvectors are orthonormal by forming the scalar product according to

$$(9) \qquad (Y_i,Y_j) = \Sigma\, Y^*_i Y_j = \delta_{ij}$$

where Y^*_i represents the complex transpose of Y_i, and δ_{ij} is the Kronecker delta. Hence step 4 has also been completed.

Now the complementary solution of Eq. (3) may be written by inspection as:

$$(10) \qquad X_c = \Sigma\, c_i X_i$$

where $i = 1,2,3$, and the c_i's are determined by the initial conditions. By performing an orthogonal transformation on Eq. (3b) the matrix, A, will be diagonalized. This results in the eigenvalue equation, with the eigenvectors Y_i corresponding to the transformed X_i. For a nondegenerate set of eigenvalues, the orthogonal transformation matrix, T, will take the form:

$$(11) \qquad T = [Y_1\; Y_2 \;.....Y_n]$$

$$(12) \qquad T\,T' = \begin{vmatrix} (Y_1,Y_1) & ......(Y_1,Y_n) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ (Y_n,Y_1) & ......(Y_n,Y_n) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0...0 \\ 0 & 1 & 0...0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & 0 & 0...1 \end{vmatrix}$$

since $(Y_i,Y_j) = \delta_{ij}$.

Upon transforming Eq. (3b) one obtains for the X_i's:

$$(13) \qquad X_i = Y_i \sin\,(\omega_i t + \phi_i)$$

where $\omega_i = (-\lambda_i)^{1/2}$ (Dettman, 1962) and ϕ_i is the initial phase angle of the ith mode.

The complementary solution, Eq. (10), now becomes:

$$(14) \qquad X_c = \Sigma\, c_i\, Y_i \sin(\omega_i t + \phi_i)$$

where c_i and ϕ_i are the arbitrary constants determined by the boundary conditions.

Next, we must find the particular solution. Let the particular solution of Eq. (3) be of the form:

$$(15) \qquad X_p = \begin{vmatrix} q_1 \\ q_2 \\ q_3 \end{vmatrix}$$

where the q_i's are constants. Then X'' is zero, and Eq. (3) becomes:

$$(16) \qquad AX_p = G\,.$$

Expanding Eq. (16) results in:

(17) $q_1 = 3a/2b; \qquad q_2 = 2a/b \text{ ; and } q_3 = 3a/2b.$

The general solution of Eq. (3) is now obtained by adding Eq. (15), with the q_i's determined, to Eq. (13). Thus:

(18) $$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = c_1 \begin{vmatrix} (1/2)^{1/2} \\ 0 \\ -(1/2)^{1/2} \end{vmatrix} \sin(\omega_1 t + \phi_1) + c_2 \begin{vmatrix} 1/2 \\ (1/2)^{1/2} \\ 1/2 \end{vmatrix} \sin(\omega_2 t + \phi_2) +$$
$$c_3 \begin{vmatrix} 1/2 \\ -(1/2)^{1/2} \\ 1/2 \end{vmatrix} \sin(\omega_3 t + \phi_3) + \begin{vmatrix} 3a/2b \\ 2a/b \\ 3a/2b \end{vmatrix}$$

It should be recalled that for the previous discussion, the masses and spring constants were set equal to constants. In the event that $m_i \neq m_j$ the matrix, A, will not be symmetric and the results are complicated considerably.

ONE-DIMENSIONAL NONUNIFORMLY ACCELERATED SYSTEMS

Consider the same mass-spring configuration as previously discussed. Let $a(t)$ represent a nonuniform acceleration of the reference frame of the system in question, once again taken as along the $+X$ axis.

The equations of motion for the nonuniformly acclerated system are:

(19) $$\begin{aligned} x''_1 &= -2bx_1 + bx_2 + a(t) \\ x''_2 &= bx_1 - 2bx_2 + bx_3 + a(t) \\ x''_3 &= bx_2 - 2bx_3 + a(t) \end{aligned}$$

where again $m_1 = m_2 = m_3$, $k_1 = k_2 = k_3 = k_4 = k$, and $b = k/m$.

The matrix form of Eq. (19) may be written as:

(20) $X'' = AX + F$

where A is identical with that of the uniformly accelerated case. Hence we know immediately that the complementary solution for the nonuniformly accelerated case will be identical with Eq. (14).

It remains to find the particular solution. Consider $a(t)$ to be such that $a(0) = 0$, $a(t_1) = 0$ and $a(t) = 0$ for $t = mt_1$, where m is an integer. This function,[4] $a(t)$, may now be expanded in a Fourier sine series with a period of $2t_1$ according to:

[4] As stated, this acceleration corresponds to an on-off, periodic perturbation with a period of $2t_1$. For an aperiodic acceleration, the solution obtained will still be valid if one allows $2t_1$ to approach an infinitely large value.

(21) $a(t) = \Sigma\ [b_m \sin\ (m\pi t/t_1)]$

where $b_m = (2/t_1) \int_0^{t_1} a(T) \sin\ (m\pi T/t_1)\ dT \ ;\ m = 1,2,3...$

Now, let $a_m(t) = b_m \sin\ (m\pi t/t_1) = [F_m]$.

Assuming a solution of the form:

(22) $X_m = Z_m \sin\ (m\pi t/t_1)$

and substituting Eq. (22) into Eq. (20) yields:

(23) $-\mu^2_m Z_m - AZ_m = B_m$

where $\mu^2_m = (m\pi/t_1)^2$; and $B = \begin{vmatrix} b_m \\ b_m \\ b_m \end{vmatrix}$

Since the set of eigenvectors of A forms a basis for the n-dimensional vector space, and Z_m is no more than a vector in an n-dimensional space, then Z_m may be expressed as a linear combination of the eigenvectors:

$$(24) \qquad Z_m = \Sigma\, \gamma_{m\alpha} Y_\alpha$$

Substitution of Eq. (24) into Eq. (23) yields:

$$(25) \qquad \gamma_{m\beta} = -(B_m, Y_\beta)/(\mu^2_m + \lambda_\beta)$$

Eq. (25) generates the γ's from which the Z_m's may be obtained. Further substitution into Eq. (22) produces X_m, which is a particular solution corresponding to an $a_m(t)$. Since this is true for every m, the complete particular solution may be found by adding the m solutions together, i.e.,

$$(26) \qquad X_p = Z_1 \sin(\pi t/t_1) + Z_2 \sin(2\pi t/t_1) + \ldots Z_m \sin(m\pi t/t_1)$$

Hence, the total solution to Eq. (20), formed by adding Eqs. (14) and (26), becomes:

$$(27) \qquad X = \Sigma_i[c_i Y_i \sin(\omega_i t + \phi_i)] + \Sigma_j[Z_j \sin(j\pi t/t_1)]$$

SUMMARY

In summary, both uniformly and non-uniformly accelerated one-dimensional oscillating systems have been treated, with the motion of the systems described by n-linear differential equations of motion. The system of equations was written in matrix representation and orthogonal transformations were performed on the configuration matrix, thus rendering it diagonal. By employing matrices, a compact form for the oscillation of the systems is obtained.

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