Linear Heat Conduction with Temperature Dependent Physical Properties[1]

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A study of transient heat conduction was initiated as an introduction to the study of heat input to a metal surface by steady illumination from an arc-image furnace or a c.w. laser. The present paper includes an analytical study for one-dimensional heat conduction from a constant heat input to a surface and of numerical solutions for: (a) a material with thermal properties that are independent of the temperature; and (b) a practical material, iron, with temperature-dependent properties. The classical case involves a differential equation with constant coefficients which gives the temperature distribution in depth as a function of the time and cannot include a solution over the temperature range which includes the heats of fusion and of vaporization. The numerical solution for iron includes the effect of temperature dependence of the specific heat, the thermal conductivity and the linear expansion over a temperature range from 300°K to the boiling point of iron at 3273°K. The classical and the numerical solutions are then compared. In addition, the numerical solution demonstrates some peculiarities of the computer solution, and the practical interpretation of these peculiarities are discussed.

BOUNDARY CONDITIONS AND EQUATIONS

The mathematical statement of this problem is consistent with the flow of heat in one dimension. Heat is assumed to be absorbed by an irradiated, flat surface at a rate of H_o per unit area. The value of H_o for the numerical examples is assumed constant and is set equal to 500 calories/cm²-sec. Heat is conducted from the irradiated face into a thin plate of thickness L and this thickness is taken as 1 cm for the numerical calculations. It is assumed that no heat is lost from the back face of the plate. The partial differential equation for this problem is,

$$\delta^2 T/\delta x^2 = (cp/k)\ \delta T/\delta t$$

The boundary condition at $x = 0$ is

$$\delta T/\delta x = H_o/k$$

and the boundary condition at $x = L$ is

$$\delta T/\delta x = 0$$

The last condition states that no heat is lost from the back face; i.e. the back face of the thin plate is insulated to prevent the flow of heat away from it. In these equations, t is the time, T is the temperature in absolute units, and x is the distance into the plate in a direction normal to the front surface where $x = 0$. The three, pertinent thermal properties for the material are: c, the specific heat; p, the density; and k, the thermal conductivity. The analytical solution for the preceding equations may be found in many sources when the physical properties are assumed constant and are entirely independent of the temperature. The recorded solution for the above equation from one source[1] (Carslaw and Jaeger, 1947) has the following form:

$$T = T_o + H_o t/cpL - H_o x/k + H_o x^2/2kL + H_o L/3k - H_o L/k \sum_{N=1}^{\infty} 2/N^2\pi^2 \exp\left(-KN^2\pi^2 t/cpL^2\right) \cos\left(N\pi x/L\right)$$

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For the purposes of numerical calculation and to extend, easily, the calculation to the same problem with different dimensions, the following substitutions are introduced.

$$t^* = t/L^2 \qquad x^* = x/L$$

The above solution may then be rewritten in the following form:

$$T = T_o + H_oL/k\,[t^*/cp - x^* + x^{*2}/2 + 1/3 - 2/\pi^2 \sum_{N=1}^{\infty} 1/N^2 \exp(-kN^2\pi^2t^*/cp) \cos(N\pi x^*)]$$

Provided the physical properties are taken to be constant, the above solution is applicable for different thicknesses of the material.

For a numerical solution to be obtained on a digital computer, it is necessary to restate the differential equation and the boundary conditions in the form of difference equations. When these relations are properly programmed for a digital computer, the problem may be solved for a heat input that is variable with time and for physical properties that are a function of the temperature. For comparison with the classical solution for constant physical properties, numerical results are obtained for an assumed heat input H_o = 500 calories/cm²-sec., a thickness of the plate L = 1 cm and an initial temperature T_o = 27°C. The physical properties are assumed to have the values that are given for 300°K in the Figs. 1, 2 and 3 for the specific heat, the density and the thermal conductivity, respectively. In order to obtain the results for different values of H_o and L, the following relations are employed.

$$x = x^*L \; ; \; t = t^* L^2$$

or,

$$T_{H_o,L} = (T_{500,1} - T_o)\, H_oL/500 + T_o$$

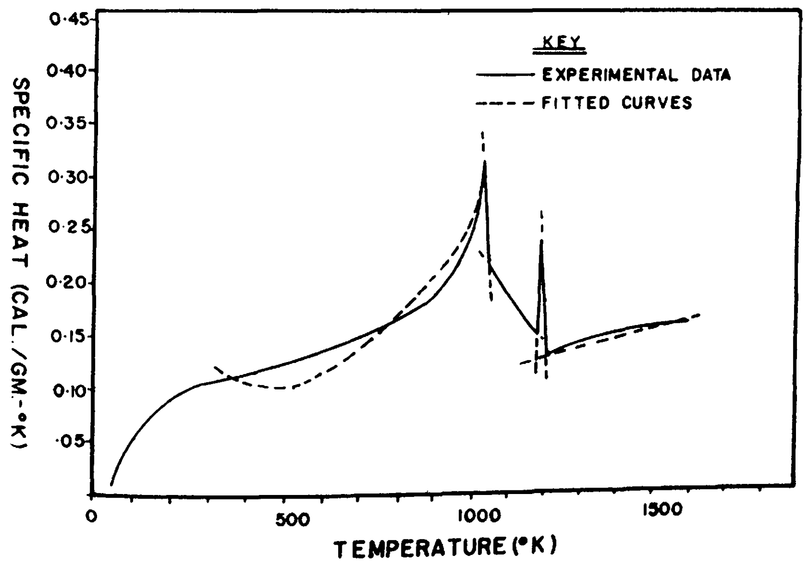

Fig. 1. Dependence of the specific heat on temperature below the melting point.

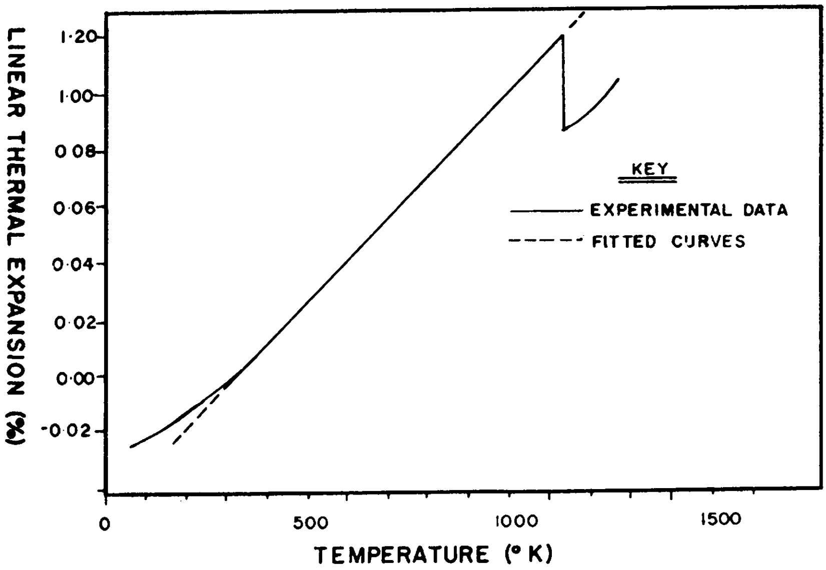

Fig. 2. Dependence of density on temperature.

The partial differential equation and the boundary conditions are replaced by the following difference equations

$$(T_{i+1,j} - 2T_{ij} + T_{i-1,j}) / \Delta x^2 = \alpha (T_{ij}) [T_{i,j+1} - T_{ij}] / \Delta t$$

where

$$\alpha(T_{ij}) = C(T_{ij})\ p\ (T_{ij})/k\ (T_{ij})$$

and at $x = L$,

$$(T_{i,j} - T_{i-1,j}) / \Delta x = 0$$

An inherent error in the differencing is introduced at the face where $x = 0$. In order to minimize this error, the third order approximation is used for the boundary condition at $x = 0$. This approximation is:

$$T_{1,j+1} = T_{1,j} + 2\Delta t/\alpha\ (T_{1,j})\ \Delta x^2\ [T_{2,j} - T_{1,j} + H_o\ \Delta x/k]$$

and to insure convergence, the following relation was employed,

$$\Delta t / \Delta x^2 = 1/6\ \alpha_{min}\ (T_{i,j})$$

where α_{min} varies in the solution and is the smallest value of α for each temperature profile. This relation insures accuracy and a rapid convergence on the solution. The size of the mesh in the solution of the problem is taken as

$$\Delta x = 0.05L = 0.05 \text{ cm}$$

which is equivalent to 20 points for calculation between the irradiated face at $x = 0$ and the insulated face at $x = 1.0$.

It is not possible completely to generalize the results of the computer solution because of the variable thermal properties. The same transformations as for the analytical solution will give approximate results. Exact results may be found if H_o and L are chosen so the temperature remains invariant upon transformation; i.e. when $H_oL/500 = 1$.

For the computer solution with difference equations the dependence of the physical properties on the temperature must be inserted in the computer. The computer program was "debugged" on an IBM 1410 with a computer memory of about 40,000 bits. The space in the memory is somewhat limited, so the temperature dependence of the physical properties is only approximated. The accuracy of the approximation to the physical constants are shown by Figs. 1, 2, and 3 for the specific heat, the density and the thermal conductivity, respectively.

With the dependence of the physical constants approximated by suitable relations, the computer program may be formulated. The program was assembled such that by changing the boundary condition routines, the program may be adapted to a different physical problem, such as a variable heat input. By a simple change in the thermal property subroutines, the solution may be adapted to a different material with physical constants that depend on the temperature in a different manner.

The program is extended through the melting temperature and to the temperature of vaporization. Since very little data exists for this region, the properties were assumed to vary linearly with the temperature. The latent heat of fusion was assumed constant. As a first approximation, these values were taken from a standard reference (Stull and Sinke, 1956).

The computer solution contains peculiarities that are inherent in the solution of difference equations instead of differential equations. In reviewing the data, the curves must always be recognized as the solution to a difference equation and not a true differential equation. There are 20 cells per cm so each cell is 0.05 cm thick. Any sharp break from a continuous curve is a consequence of the solution of a difference equation by the computer.

The solutions from the computer, with temperature dependent thermal properties, are compared with the analytical solution when the thermal properties are constant. The values of the physical constants in the analytical solution are taken as the value of the curves for the physical con-

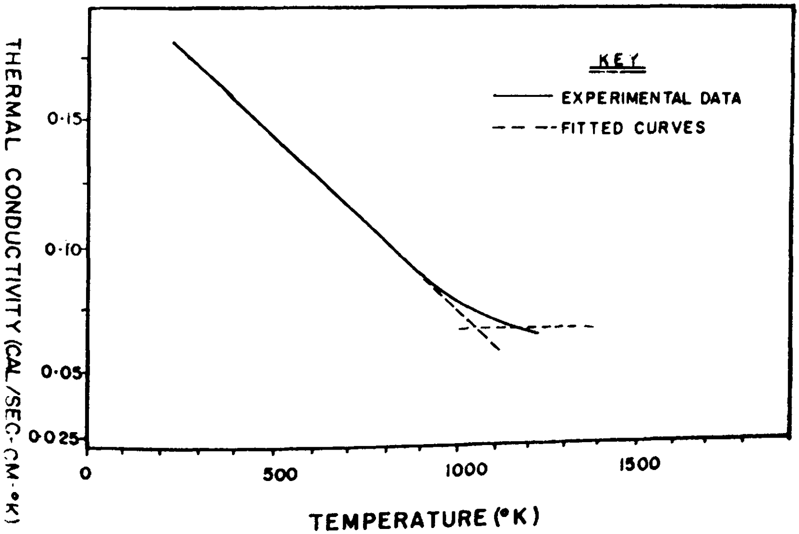

Fig. 3. Dependence of thermal conductivity on the temperature.

stants that are represented by the dotted lines on Figs. 1, 2, and 3. Fig. 4 represents the curves for the temperature distribution through the plate 0.138 secs after the heat first starts to enter the plate; the computer and the analytical solution can be compared. The surface temperature is found to be higher for the computer solution with temperature-dependent coefficients.

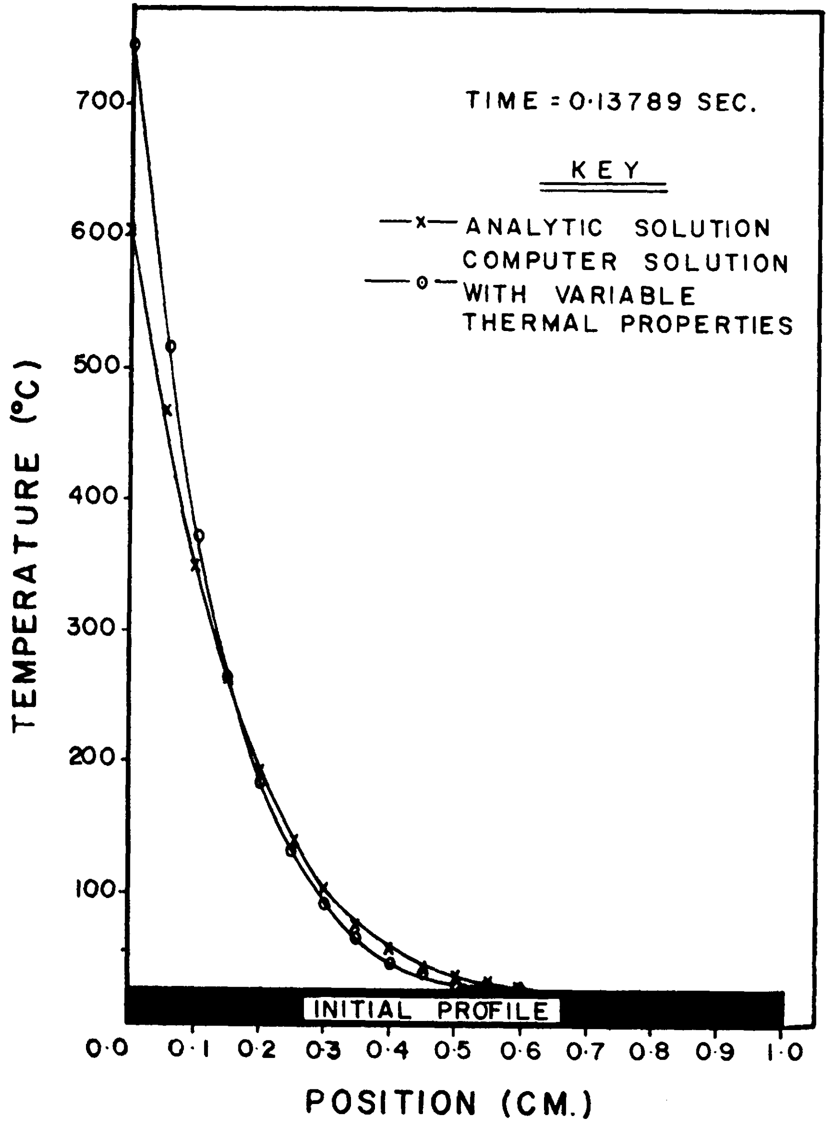

Fig. 4. Comparison of calculated temperature distribution through the plate by the computer solution (temperature-dependent constants) and by the analytical solution (temperature-independent constants). Time 0.138 seconds.

As time progresses, the results of the computer solution for surface temperature becomes still higher than those of the analytical solution. This is illustrated in Fig. 5 for a time 0.459 sec after time zero. The transformation from the magnetic to the non-magnetic forms of iron occurs between 1000 to 1250°K, and the first opportunity to observe the effects on the transformation is in Fig. 5. There does not, however, appear to be any significant change in the trend for the surface temperature to

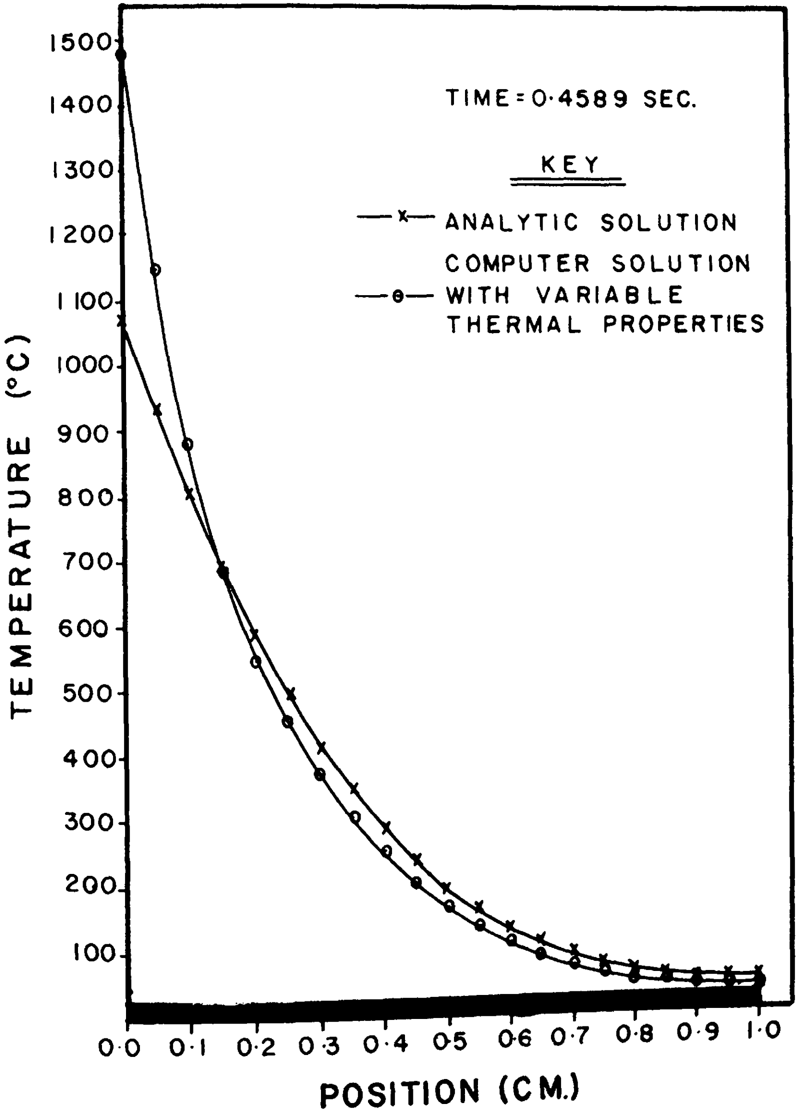

Fig. 5. Comparison of calculated temperature distribution by the computer and by the analytical solution. Time 0.459 seconds.

become much higher for the computer solution than for the analytical solution. This is attributable, primarily, to a rapid change in the density which accompanies the short range increase in the specific heat.

As the heat input continues, the surface temperature rises past the heat of fusion and approaches the boiling point of iron. The analytical solution becomes too involved for practical results in this temperature range. The curves in Fig. 6 show the temperatures of the first five cells as a function of time. The cells are numbered from the surface, which is Cell 1, toward the back of the plate; there are 20 cells in the plate of one-centimeter thickness. The solid lines show the temperatures calculated by the computer. An estimated surface temperature is indicated by the dotted line, which is designated $x = 0$. The humps and irregularities in these curves do not correspond to actual results but are introduced by the computer.

The use of difference equations to replace the differential equations for a computer solution introduces results that do not exist in reality. A startling example of this are the horizontal segments in Fig. 6; the horizontal sections seen in all the solid lines simply do not exist in fact, likewise, the apparent pulses at the temperature above the fusion point for Cell 1 are induced by the horizontal sections in Cells 2, 3, 4, etc., and do not exist either. An estimate of the surface temperature in the physical problem is indicated by the dotted line. The horizontal sections in the temperature curve for each cell are a consequence of the finite size of the meshes; that is why they have no parallel in the physical situation. The flat sections occur when the computer treats each cell as a unit. The temperature remains constant until sufficient energy flows into a mesh, at constant temperature, to supply the heat of fusion for the entire volume of the mesh. Then the heat must flow into the next mesh until it also has received the latent heat of fusion. While the heat is flowing into the second mesh, there is an inflection in the surface temperature, as indicated by the curve for the temperature of Cell 1. Actually, the metal melts in infinitely thin layers which produce a change in slope of the curve, but not a flat place such as indicated in Fig. 6.

In the computer solution, the time that a cell remains at a constant temperature increases with depth of the cell below the surface. This is expected for a constant rate of heat input which increases the surface temperature. Some heat is required to increase the temperature near the surface and the fraction of the input heat that reaches a cell decreases with depth below the surface. A longer time is then required to melt the same volume of material. It should be noted that the flat sections will decrease in length and the solution will approach the true solution as the size of the mesh for the computer solution is decreased.

CONCLUSIONS

The classical analytical solution cannot be extended by elementary mathematical techniques over a temperature range that includes the latent heat of fusion and the latent heat of evaporation. An analytical solution for the surface temperature as a function of time after the start of the heat input was found with constant coefficients in the differential equation for temperatures up to 1100°K. This solution is shown by the lower curve in Fig. 7 when the coefficients were taken as constant at their values for 300K. The comparable numerical solution with temperature dependent coefficients is shown by the upper curve in Fig. 7. The difference is quite significant. To show that the difference does not arise from the computer, as it obtains the numerical solution, the lower curve for the analytical solution was checked by obtaining a numerical solution with the computer. The results of the computer solution converged rapidly to those that were predicted by the classical, analytical solution.

The temperature dependence of the coefficient in the differential equation for heat conduction must be considered in order to obtain accurate results. This is true for temperatures below the melting point. The only apparent difficulty with the numerical solution above the melting point for a change of phase is the discontinuity that occurs at each change of phase, even when the change of phase is a small one, as at 1000 and 1200K in the solid state. This paper shows that the effect of the discontinuity can be minimized to any desired degree by a decrease in the size of the cells that are considered for the numerical solution.

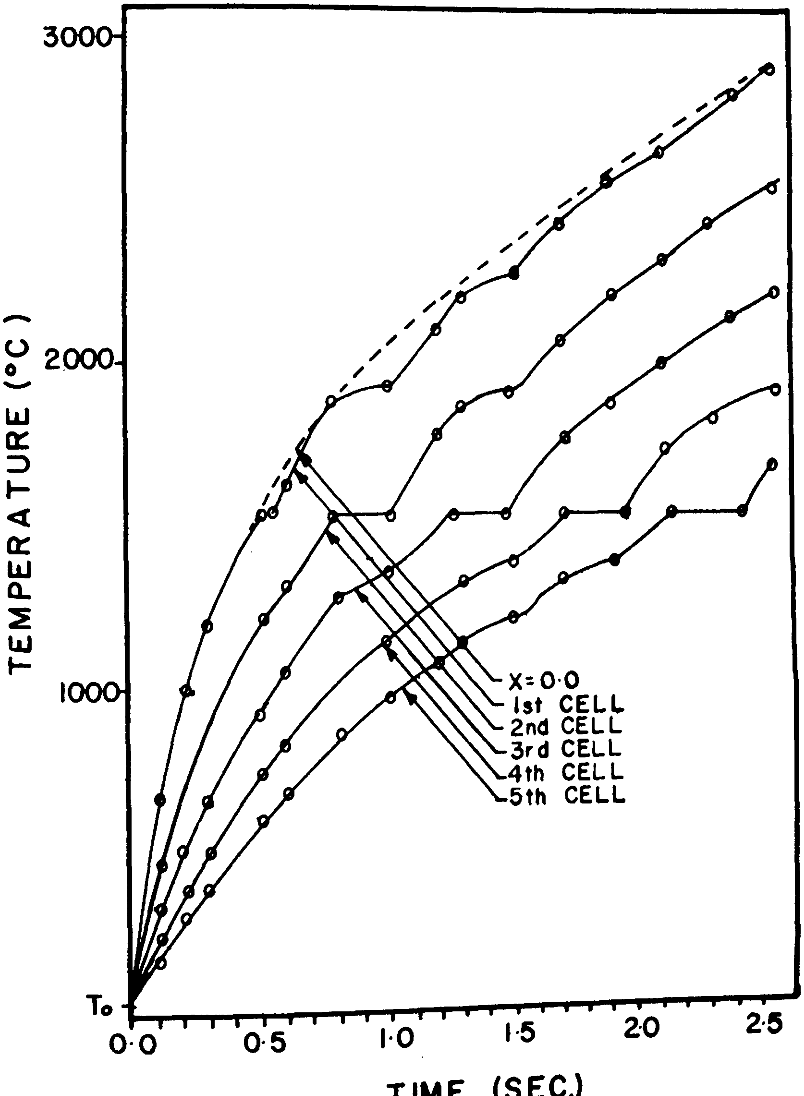

Fig. 6. Computer solution for the time dependence of the temperature of the first five of the twenty cells in a one centimeter thick plate. Estimated true surface temperature represented by a dotted line.

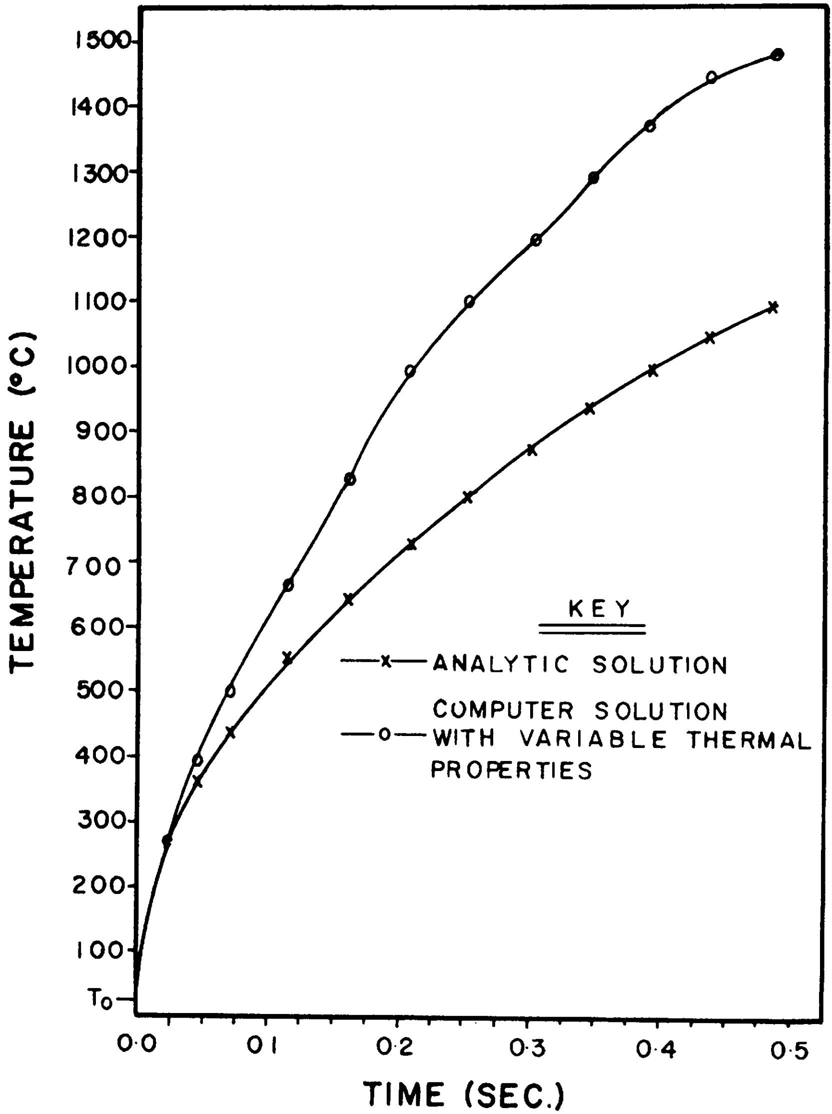

Fig. 7. Surface temperature dependence on time according to computer and analytical solutions.

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