SECTION D, SOCIAL SCIENCES

The Hydrostatics of Archimedes

BETTY RUTH ESTES, University of Oklahoma, Norman

The study of hydrostatics made by Archimedes of Syracuse (*ca.* 287-212 B.C.) is usually mentioned in connection with the so-called Archimedean principle, that a body immersed in a fluid will be lighter by the weight of the fluid displaced. Archimedes gave a mathematical development of his principle in Book I of a work called *On Floating Bodies*. Much attention has been paid to his first book on hydrostatics, but Book II of *On Floating Bodies* has been virtually ignored. The second book, nevertheless, better indicates the mathematical genius of Archimedes.

There are excellent reasons for the preference for Book I over Book II. Book I is short, containing only two postulates and nine propositions, but its contents are important in the later development of hydrostatic theory. Archimedes was able to provide the proof of his principle by using only the first postulate and seven theorems.[1] The postulate is: in a fluid whose parts are at the same level and are continuous, a part which is under less pressure will be pushed away by an adjacent part which is under more pressure. Furthermore, each part of the fluid will be "pressed by the fluid which is vertically above it, if the fluid is not shut up in anything and is not compressed by anything else."[2] After stating the first postulate, Archimedes demonstrated that the surface of any fluid at rest is a sphere with its center at the center of the earth. Using this theorem and the postulate, he gave proofs of five propositions dealing with what happens when bodies of density equal to, greater than, or less than that of the fluid are placed in the fluid. The methods of proof require elementary geometrical knowledge. The first part of Book I is an elegant, self-contained work.[3]

In contrast to the general applicability of most of the first book, the second book treats the specialized topic of a right segment of a paraboloid of revolution floating in a fluid (the paraboloid is formed by revolving a parabola about its axis). The problem which Archimedes investigated is: what are the conditions for stable equilibrium of the paraboloid in the fluid, given varying lengths of the paraboloidal axis and varying ratios of the specific gravities of the two substances? The paraboloid is always assumed to be less dense than the fluid, since it is only partly submerged.

If the paraboloid is wide enough in relation to its depth, the answer can be expressed simply. Suppose that the length of the axis is less than or equal to three-fourths of the latus rectum. Archimedes determined that for such a paraboloid, if the base is completely submerged or if the base lies completely above the surface of the fluid, there is stable equilibrium only when the axis is vertical. This is independent of the ratio of the specific gravity of the paraboloid to the specific gravity of the fluid.

There is a much more complex situation when the length of the axis is greater than three-fourths of the latus rectum. The position of equilibrium is then dependent upon the ratio of the specific gravities. For a ratio larger than a value determined by Archimedes, there is equilibrium with a vertical axis when the base lies above the fluid; for the submerged base, there is stable vertical equilibrium if the ratio is sufficiently small. Archimedes also found that, if the ratio of the axis length to the latus rectum were greater than a given value, he could distinguish six different, specified positions of stable equilibrium, corresponding to six sets of values for the ratio of specific gravities. In this case, the base was originally entirely above the surface of the fluid.[4]

The proofs of the propositions in Book II differ considerably from those in Book I. Archimedes used the results from the first book, but he did not specifically employ the first postulate in the second book. He could not use it because it follows from the first postulate that all vertical lines converge at the center of the earth . However, the proofs in Book II rest on the assumption that the surface of the fluid is a plane and that the verticals are parallel.[5] Archimedes used this assumption in connection with the second postulate of Book I, which states that, if a body is forced upward in a fluid, the force is along a line through the center of gravity of the body and perpendicular to the surface of the fluid.[6]

An examination of one of the simpler propositions indicates the method applied by Archimedes in these proofs. Consider the case when the length of the axis is less than or equal to three-fourths of the latus rectum, and the axis of the paraboloid is tilted from the vertical so that the base remains entirely above the surface. Archimedes proved that the paraboloid would return to the vertical position in the following manner: Using theorems from other works, he could find the centers of gravity of the whole segment and of the submerged segment. He was able to show that the force acting on the center of gravity of the submerged portion pushed upward on that part of the paraboloid which tilted down. Central to his argument was a geometric demonstration of the relative positions of the centers of gravity of the submerged and the total segments. Archimedes also knew that the center of gravity of the portion above the fluid lay on the line joining the other two centers. Since the vertical force on the center of gravity of the upper portion acted downward, the paraboloid turned until the three centers of gravity all lay on the vertical axis.[7]

When the ratio of the specific gravities was included in the problem, another element was added to the method of proof. Archimedes applied the first proposition of Book II, according to which the specific gravity of the paraboloid is to that of the fluid as the immersed portion of the solid is to the whole. But, by a theorem in another work, these two segments of the paraboloid are to each other as the squares of their axes. This gave Archimedes an expression with which he could make the essential determination of the relative positions of the two centers of gravity, as in the previous example.[8]

The proofs of Book II involve geometrical demonstrations far above the level of those in Book I. Furthermore, it is not immediately evident how Archimedes might have arrived at the statements in the second book, but a difficult mathematical analysis must have been involved. Thus, Book II of *On Floating Bodies* is mathematically a more original work than the first part of Book I. Even though Archimedes has been called the founder of hydrostatics because of his work in the first book, he perhaps had precursors for it. His primary contribution in the first book may have been to take ideas which were already developed and to choose the proper postulate from which he could develop their proofs.[9] In Book II, however, Archimedes brought to bear his knowledge in many areas of mathematics. He incorporated theorems from his own writings on centers of gravity, on conoids and spheroids, on the quadrature of the parabola, and perhaps from some unknown works. In view of the background that was necessary for the writing of Book II, it is improbable that this work could have been conceived or carried out by any mathematician before Archimedes.

It is also possible that Book II better represents the interests of Archimedes. It has been noted that Book I resembles a textbook, whereas Book II contains the type of specialized mathematical investigation which was typical of Archimedes.[10] But, no matter which of the two books interested Archimedes most, the neglected work on hydrostatics deserves more attention than it has received, both because of its originality and the mathematical ability demonstrated in it.

NOTES

[1]The second postulate, and propositions eight and nine, are more closely related to the work in Book II than to the major part of Book I.

[2]E. J. Dijksterhuis, *Archimedes*, trans. C. Dikshoorn, Vol. XII of *Acta historica scientiarum naturalium et medicinalium* (Copenhagen: Ejnar Munksgaard, 1956), p. 373. Compare with T. L. Heath's translation in *The Works of Archimedes. Edited in Modern Notation with Introductory Chapters by T. L. Heath. With a Supplement, The Method of Archimedes, Recently Discovered by Heiberg* (New York: Dover Publications, Inc., [195-?]), p. 253. Heath's translation will be referred to hereafter as Archimedes, *Works*.

[3]Archimedes, *Works*, pp. 253-62; Dijksterhuis, *Archimedes*, pp. 373-77.

[4]Archimedes, *Works*, pp. 263-300; Dijksterhuis, *Archimedes*, pp. 380-98.

[5]Dijksterhuis, *Archimedes*, pp. 377-79; Paul Tannery, "Sur l'histoire de la pression hydrostatique. Note inédite de Paul Tannery," *Archeion*, XIX (1937), 68.

[6]Archimedes, *Works*, p. 261; Dijksterhuis, *Archimedes*, p. 381.

[7]Archimedes, *Works*, pp. 264-66; Dijksterhuis, *Archimedes*, pp. 381-85.

[8]Archimedes, *Works*, pp. 266-300; Dijksterhuis, *Archimedes*, pp. 385-97.

[9]Jean Daujat, "Note sur les origines de l'hydrostatique," *Thales*, I (1935), 55; compare with the remarks of P. Duhem in *Les origines de la statique* (2 vols. bound in 1; Paris: Librairie Scientifique A. Hermann, 1905-1906), II, 279-80.

[10]T. L. Heath, "Introduction," in Archimedes, *Works*, p. xl.

REFERENCES

Archimedes. *The Works of Archimedes. Edited in Modern Notation with Introductory Chapters by T. L. Heath. With a Supplement, The Method of Archimedes, Recently Discovered by Heiberg.* New York: Dover Publications, Inc., [195-?].

Daujat, Jean. "Note sur les origines de l'hydrostatique," *Thales*, I (1935), 53-56.

Dijksterhuis, E. J. *Archimedes*. Translated by C. Dikshoorn. Vol. XII of *Acta historica scientiarum naturalium et medicinalium*. Copenhagen: Ejnar Munksgaard, 1956.

Duhem, P. *Les origines de la statique*. 2 vols. bound in 1. Paris: Librairie Scientifique A. Hermann, 1905-1906.

Tannery, Paul. "Sur l'histoire de la pression hydrostatique. Note inédite de Paul Tannery," *Archeion*, XIX (1937), 67-69.