A Hydrodynamical Treatment of the Electrically Driven Shock Tube[1]

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In the past ten years, successive researches directed by R. G. Fowler at the University of Oklahoma have developed an electrically driven shock tube (Fig. 1), a tool for investigation of fluid flow situations which cannot be readily observed with shock tubes of the standard design (Fig. 1). Although the fluid flow observed in the electrically driven tube can be compared to that observed in the conventional tube, a precise analogy is not applicable. Perhaps one explanation for the variance between the two tubes may be found in the analysis of the thermodynamic nature of the rarefaction wave which moves into the volume of gas between the energy input electrodes of the electric tube. The purpose of this paper is to analyze this additional possibility for the flow processes in the electrically driven tube.

The shock tube of conventional design consists of a compression chamber separated by a mechanical diaphragm from an expansion chamber at a relatively lower initial pressure. At time t_0 the diaphragm is ruptured causing a compressive wave to travel into the expansion chamber. At some later time t_1, the compressive wave has steepened into a shock discontinuity moving with ultra-acoustical speed and raising the pressure to p_1, an intermediate value between p_0 (the initial pressure in the expansion chamber) and p_3 (the initial pressure of the compression chamber). At some distance behind the shock wave there is a contact surface separating the gases that were initially in the separate chambers. Pressure and flow velocity are equal across this contact surface, although the density and temperature may differ. Region 2 is connected to the undisturbed gas in the compression chamber, region 3, by a rearward facing rarefaction wave. The time period of interest here does not include the conditions following reflection of either wave at the ends of the tube.

The electrically driven tube may be compared hydrodynamically to the idealized conventional tube with the acceptance of two assumptions. First, it is assumed that at time t_0 the capacitor is instantaneously discharged into the compression chamber between the electrodes raising the temperature and pressure in this region to equilibrium at T_3 and p_3. Secondly, it is assumed that only hydrodynamic forces are in effect and the electrodynamic forces and flows present as a result of the creation of a plasma may be neglected.

On the basis of these assumptions regarding the applicability of the hydrodynamic description of flow on the electric tube, an expression for the shock Mach number M may be derived as shown in Fig. 2. Equations (1), (2), and (3) express the conservation of mass, momentum, and energy for an ideal un-ionized gas contained in a cylinder of unit cross-section which passes through a stationary plane shock wave.

Transformation of the coordinates to the laboratory system in which the gas up-stream of the shock in stationary is accomplished by equations (4) and (5). Algebraic manipulation of (1), (2), (3), (4) and (5) yields the relations (6) and (7).

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[1] Supported in part by the U.S. Office of Naval Research.

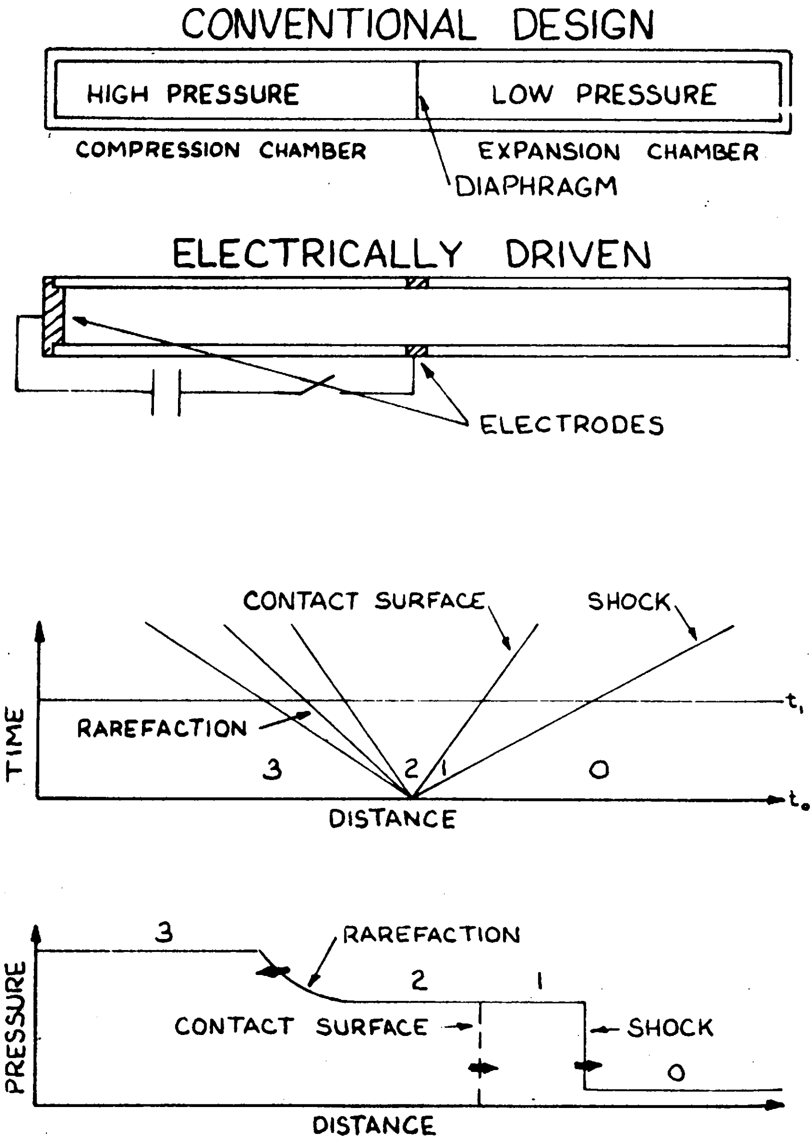

FIGURE 1 THE SHOCK TUBES

The rarefaction wave under question may be described by consideration of the differential equations of fluid dynamics which govern continuous motion of a homogeneous medium. For the case of an inviscid elastic fluid in one dimensional flow the differential equations applicable for the conservation of mass and momentum are equations (8) and (9) in Fig. 3.

(1) $$\rho_o v_o = \rho_1 v_1$$

(2) $$p_o + \rho_o v_o^2 = p_1 + \rho_1 v_1^2$$

(3) $$\tfrac{1}{2} v_o^2 + c_p T_o = \tfrac{1}{2} v_1^2 + c_p T_1$$

ρ = density

v = fluid velocity relative to the shock wave

c_p = specific heat per unit mass at constant pressure

T = absolute temperature

Subscripts o and 1 refer to the states in the regions ahead and behind the shock wave as noted in Fig. 1.

(4) $$v_o = U$$

(5) $$v_1 = U - u_1$$

U = shock velocity

u_1 = particle velocity behind the shock

(6) $$u_1 = \frac{U(\mu - 1)(1 - \xi)}{\mu + \xi}$$

(7) $$\frac{U}{a_o} = \left[\frac{(\mu + \xi)}{\xi(\mu + 1)}\right]^{\frac{1}{2}}$$

a_o = "sound speed" in the region ahead of the shock, $a = \left(\frac{dp}{d\rho}\right)_s$

μ = function of the specific heats, $\mu = c_p + c_v / c_p - c_v$

ξ = pressure ratio across the shock,
$$\xi = \frac{p_o}{p_1} = \frac{p_o}{p_2}$$

FIGURE 2

These differential equations may be solved explicitly, knowing the appropriate boundary conditions, by a method involving characteristics, provided a third equation such as (10) or (11) is specified.

Equation (10) is applicable if the rarefaction takes place adiabatically. Equation (11) is appropriate if an isothermal expansion is assumed.

Let it be assumed that the rarefaction takes place isothermally. This is a possibility due to high heat conduction as a result of the presence in this region of high velocity electrons in the plasma. The theory of characteristic solutions plus the boundary condition $u_1 = 0$ yields relation (12) across the rarefaction.

Equation (12) together with the specifying equation (11), the fact that $u_1 = u_2$, and the definition of sound speed, may be brought into the form of equation (13), which when solved with (6), gives equation (14).

Equation (14) may now be solved with (7) to obtain M as a function of K_3/a_o. For a monatomic gas, this solution reduces to equation (15). Equation (15) is presented in Fig. 4 for value of M between 3 and 50.

(8) $\rho u_x + u\rho_x + \rho_t = 0$

subscripts t and x are the customary notations for partial differentiation with respect to time and distance.

(9) $\rho u u_x + \rho u_t + p_x = 0$

(10) $p/\rho^{\gamma} = \text{constant}$

γ = ratio of specific heats c_p/c_v

(11) $p/\rho = \text{constant}$

(12) $K_3 \ln\rho_3 = u_2 + K_3 \ln\rho_2$

$K_3 = \frac{(1+\alpha_3)}{m} k T_3$

α_3 = degree of ionization in region 3

k = Boltzman's constant

m = mass per atom

(13) $\frac{u_1}{a_o} = \frac{K_3}{a_o} \ln\left[\gamma \left(\frac{K_3}{a_o}\right)^2 \xi\right]$

M = shock Mach number $\frac{U}{a_o}$

(14) $M = \frac{\mu+\xi}{(\mu-1)(1-\xi)} \frac{K_3}{a_o} \ln\left[\gamma \left(\frac{K_3}{a_o}\right)^2 \xi\right]$

(15) $\frac{3(M^2-1)}{4M} = \frac{K_3}{a_o}\left[\ln 20 \left(\frac{K_3}{a_o}\right)^2 - \ln(15M^2-3)\right]$

FIGURE 3

Let it now be assumed that the rarefaction takes place adiabatically as is common in the analysis of the conventional tube. By the same method as above one finds the equations of Fig. 5. Equation (16) is valid across the rarefaction, additional substitution with the appropriate equations as before yields equation (17), and this is solved with (7) to again find M as a function of K_3/a_o, equation (18). Equation (18) is presented in Fig. 4 for values of M between 3 and 50.

The energy input to the electric tube is readily obtained experimentally. The same is true for the velocity of the shock front. The validity of the preceding assumptions and results could be appraised if the energy input is associated with the parameter K_3 and thereby with the shock Mach number. The total specific energy of the gas initially in the compression chamber may be found from equation (19) or, in terms of K_3, from equation (20), Fig. 5.

Thus the theoretical energy required to produce a hydrodynamic shock wave of a given speed may be determined with knowledge of the type and initial state of the gas plus appropriate assumptions regarding the degree of ionization and the thermodynamic nature of the rarefaction wave.

In order to observe the form of the solutions of equation (20) Fig. 6 has been prepared using argon as the shocked gas, T_s=300°K, I=15.68 ev, the first ionization potential, and obtaining the values for K_3/a_0 from Fig. 4.

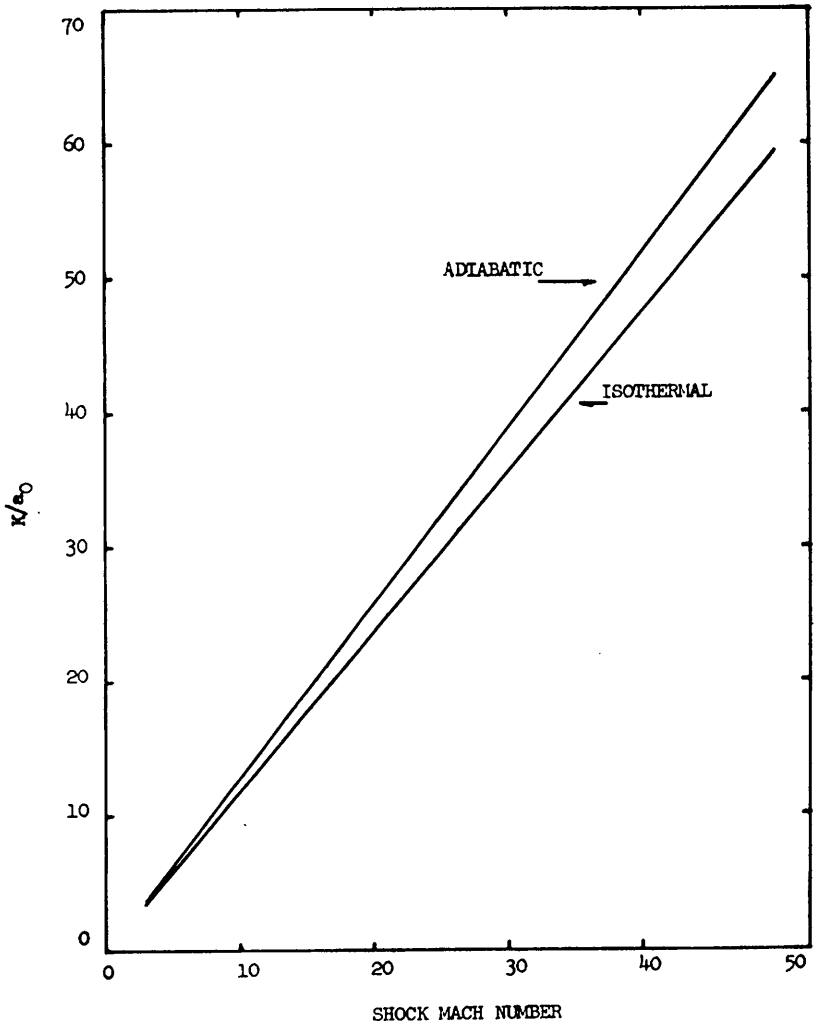

FIGURE 4 THE VARIATION OF K_3/a_0 WITH SHOCK MACH NUMBER

It is apparent that with both thermodynamic assumptions for no ionization at all shock speeds and for total ionization at higher speeds the specific energy input is proportional to the square of the shock velocity. This is not an unfamiliar relationship.

The experimenter may desire to verify or disprove the hypothesis that the electrically driven tube may be analyzed assuming only hydrodynamic flow. If so, the choice of the thermodynamic treatment of the rarefaction wave appears to be unimportant, as the differences of the energy-shock velocity relationships are quite small from his viewpoint.

(16) $$\frac{u_2}{2} + \frac{a_2}{\gamma - 1} = \frac{a_3}{\gamma - 1}$$

(17) $$M = \frac{\mu + \xi}{(\mu - 1)(1 - \xi)} \left[\gamma^{\frac{1}{2}} \frac{2}{\gamma - 1} \left(\frac{K_3}{a_o}\right) - \gamma^{\frac{1}{2\gamma}} \frac{2}{\gamma - 1} \left(\frac{K_3}{a_o}\right)^{\frac{1}{\gamma}} \xi^{\frac{1-\gamma}{2\gamma}} \right]$$

(18) $$\frac{0.194\ (M^2 - 1)}{M} = \frac{K_3}{a_o} - 0.684\ (5M^2 - 1)^{0.2} \left(\frac{K_3}{a_o}\right)^{0.6}$$

(19) $$W_3 = \frac{3}{2}\ \frac{(1 + \alpha_3) k\ T_3}{m} + \frac{\alpha_3\ I}{m}$$

(20) $$W_3 = \frac{3}{2}\ K_3^2 + \frac{\alpha_3\ I}{m}$$

FIGURE 5

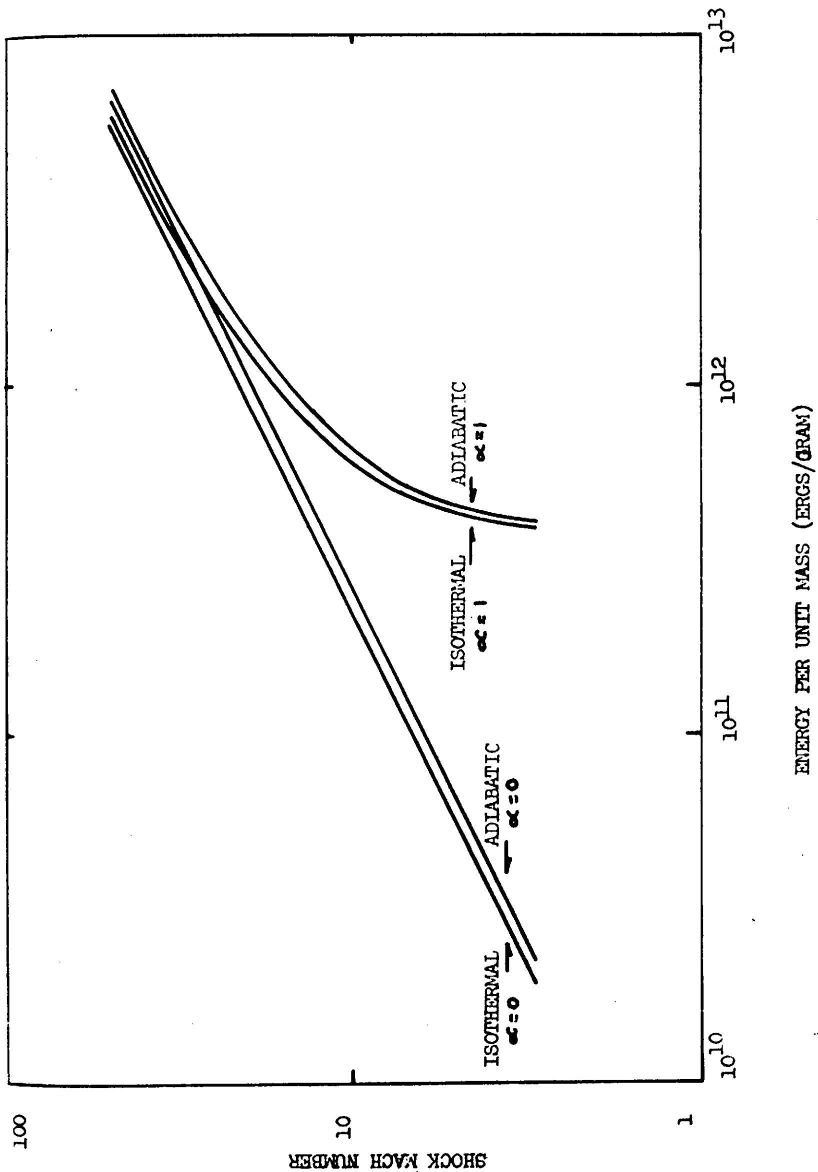

FIGURE 6 THE VARIATION OF SHOCK MACH NUMBER WITH SPECIFIC ENERGY INPUT