Dimensional Analysis and Scaling Ratios for Flow in a Porous System

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This study was made in order to give another critical examination to a method (4) of scaling certain dimensionless ratios with the point of view of finding an alternative method. The study also included a similar method (1) and relates directly to a current experimental project on flow in porous media. To provide project engineers a concise summary of certain aspects of dimensional analysis particularly important in the present problem, the details of the study made are preceded by the following discussion.

The alternative method selected was the classical π-Theorem, so far as it is possible to apply that Theorem with addition of the principle of dynamical similarity. This method differs at least in detail from the method of Leverett, *et al.* (4) in that it uses ratios of numbers representing physical magnitudes directly entering the problem, while the previous method scales only the ratios of the *dimensions* of the physical quantities. It would seem *à priori* that a method scaling numbers representing the physical quantities would be preferable; however, the present study shows that both methods lead to the same numerical values of the ratios. Aside from this, the use of the π-Theorem has other distinct advantages, chiefly in explicitly displaying certain important assumptions and conclusions seemingly not deducible from the other method.

At least one assumption is common to both approaches to scale ratios. Experimental information must be so complete that all the variables pertinent to the problem are known, including dimensional constants. In the π-Theorem it is also assumed that there exists just one functional relation between the variables. This function must be "complete" in the sense that its form must not be changed if the sizes of the fundamental units are changed, e.g., ft./sec. to mi./hr. In the present problem the form of the assumed functional relation is not known, and the π-Theorem does not necessarily help to fix the form; but it will place the arguments, i.e., the pertinent variables, in certain forms and tends to minimize the number of independent quantities which must be considered. Thus, the pertinent variables $a_1, a_2, \ldots, a_n$ in the function $\phi(a_1, a_2, \ldots, a_n) = 0$ can always be rearranged so as to give $F(\pi_1, \pi_2, \ldots, \pi_m) = 0$. The π's are dimensionless combinations of a's (products, powers, ratios). If there are r independent or primary a's, then $m = n - r$ is the number of π's to be found. As noted below it is highly important that experimental information be adequate to indicate the greatest possible number of independent quantities, since this leads to the most explicit functional relation which can be found by application of the π-Theorem.

In many problems it may be of little importance to change the form of the arguments. In the present study its importance is that it is the easiest way to an application of the principle of dynamical similarity, sometimes called a corollary of the π-Theorem. Details of this principle will be given in the examples below. Two systems are similar if the various pertinent variables are proportional, such as $a_2 = \alpha a_1$, $b_2 = \beta b_1$ and so on. The systems then behave similarly provided some function of these variables has the same set of values for either system.

The problem of whether a dimensional constant should be mysteriously inserted into the final formula is solved by the following statement: dimensional constants enter in the same manner as any other pertinent variable. However, some unusual dimensional constant may be required if some peculiar system of units is adopted, e.g., if the unit of mass is the sum of the various masses entering the problem. The ordinary systems of mechanical units do not require such handling.

Another important aspect of the π-Theorem is that one may get seemingly different results according as different numbers of independent variables are assumed. Ordinarily one will infer that the number of independent pertinent variables equals the (minimum) number of fundamental dimensions (M, L, T, for example) required for the dimensional description of the pertinent variables. Results of this policy are not untrue but usually are the most general form of possible results of the π-Theorem. In many problems additional independent variables can be identified. As an example, a problem may lead to $F(\pi_1, \pi_2) = 0$, or $\pi_1 = f(\pi_2)$, where the functional form will be unknown. However, in the same problem by physical reasoning one may be able to deduce that another of the pertinent variables, e.g., force, is an independent quantity. This assumption will reduce the number of π's to one, so that we must have $\pi =$ constant. Thus, much more specific information may become available (2). In the latter case, only one physical experiment may be sufficient to establish the value of the constant. There is always a particular form of the function $f(\pi_2)$ which will reduce $\pi_1 = f(\pi_2)$ to the form $\pi =$ const.

If data from model experiments can suggest the form of the unknown function, the functional relation can then be checked by application of the formula either to the full-scale physical situation or to another model situation having appreciably different physical size, perhaps, but which is otherwise dynamcially similar to the first model.

The present treatment of the problem assumes that M, L, T are the only independent basic quantities. There follows a detailed application of the π-Theorem with use of the pertinent variables chosen by Leverett *et al.* (4):

g = gravitational acceleration; LT^{-2}
B_o = fluid conductivity; $M^{-1} L^3 T$,
P_c = capillary pressure; $ML^{-1} T^{-2}$,
Q_t = quantity of flow; $L^3 T^{-1}$,
q_t = flow rate; $L T^{-1}$,
L = typical linear dimension; L,
θ = time; T,
$\Delta\rho$ = mass density; $M L^{-3}$.

The subscripts zero refer to the oil phase. Of course, only those pertinent variables having dimension are taken. Thus, as noted above, there will be five dimensionless π's which result from the method. Incidentally, these π's are not unique, but only are independent of each other. For example, one may be able to take the ratio of two π's and then replace one of them with the ratio. If the new set of π's so generated still contains five independent quantities, it may replace the first set. Another fact of some use in selecting the π's is that the set can contain the obvious ratios which connect some of the assumed pertinent variables. For example, see π_3 below.

Now each π is the product of the eight pertinent variables:

$$[\pi] = g^s B_o^t P_c^u Q_t^v q_t^w L^x \theta^y \Delta\rho^z$$
$$= (LT^{-2})^s (M^{-1}L^3T)^t (ML^{-1}T^{-2})^u (L^3T^{-1})^v (LT^{-1})^w L^x T^y (ML^{-3})^z.$$

Since each π is dimensionless, we may equate the combined exponents of M, of L, and of T respectively to zero:

$$\begin{aligned} -t + u + z &= 0 \\ s + 3t - u + 3v + w + x - 3z &= 0 \\ -2s + t - 2u - v - w + y &= 0 \end{aligned} \qquad (1)$$

These equations may be solved[1], perhaps, for three variables in terms of the other five variables. By inspection of the coefficients of s, t, u it is seen that the coefficient determinant does not vanish, and hence we find:

$$\begin{aligned} 3s &= v - w + x + 2y + 2z \\ 3t &= -5v - w - 2x - y + 2z \\ 3u &= -5v - w - 2x - y - z \end{aligned} \qquad (2)$$

Here the variables (v, w, x, y, z) are arbitrary and we may select any five independent solutions, such as (1, 1, 0, 0, 0), (0, 1, 1, 0, 0), (0, 1, —1, 1, 0), (0, 2, 0, 0, 1), (1, 0, 0, 0, 1). These are known to be independent because the rank of their matrix is 5. The independence of the π's obtained may be verified by noting that each will contain a pair of variables which no other π contains.

The corresponding values of (s, t, u) are (0, —2, —2), (0, —1, —1), (0, 0, 0), (0, 0, —1), (1, —1, —2) respectively. Therefore, one set of five dimensionless ratios is:

[1] Any such homogeneous set of n linear equations has solutions, but not necessarily for every arbitrary set of n variables in terms of all the others. Here the three equations are independent of each other and hence define some three variables in terms of the other five, which then are arbitrary.

$$\pi_1 = \frac{Q_t \quad q_t}{B_o^2 \; P_c^2}, \quad \pi_2 = \frac{Lq_t}{B_o \; P_c}, \quad \pi_3 = \frac{q_t \; \theta}{L},$$
$$\pi_4 = \frac{q_t^2 \; \Delta\rho}{P_c}, \quad \pi_5 = \frac{Q_t \; \Delta\rho \; g}{B_o \; P_c^2}. \tag{3}$$

The π-Theorem then indicates that $f(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = O$. Under certain hypotheses such a function may be solved for one variable, say

$$\pi_1 = F\,(\pi_2, \pi_3, \pi_4, \pi_5). \tag{4}$$

If Eq. 4 is taken to represent the prototype which is being simulated by a model which is represented by:

$$\pi'_1 = F\,(\pi'_2, \pi'_3, \pi'_4, \pi'_5), \qquad (4)'$$

then we may apply the principle of dynamical similarity. The two systems are said to be dynamically similar when corresponding points may be established such that the function F has the same value in Eq. 4 as in Eq. 4', and such that:

$$\pi_1 = \pi'_1, \; \pi_2 = \pi'_2, \; \ldots, \pi_5 = \pi'_5.$$

$$\frac{Q_t}{Q_t'} \cdot \frac{q_t}{q_t'} = \left\{\frac{B_o}{B_o'}\right\}^2 \cdot \left\{\frac{P_c}{P_c'}\right\}^2 \quad (5), \qquad \frac{L}{L'} \cdot \frac{q_t}{q_t'} = \frac{B_o}{B_o'} \cdot \frac{P_c}{P_c'} \quad (6),$$

$$\frac{L}{L'} \cdot \frac{\theta}{\theta'} \cdot \frac{g}{g'} = \frac{B_o}{B_o'} \cdot \frac{P_c}{P_c'} \quad (7), \qquad \left\{\frac{q_t}{q_t'}\right\}^2 \cdot \frac{\Delta\rho}{\Delta\rho'} = \frac{P_c}{P_c'} \quad (8).$$

$$\frac{Q_t}{Q_t'} \cdot \frac{\Delta\rho}{\Delta\rho'} \cdot \frac{g}{g'} = \frac{B_o}{B_o'} \left\{\frac{P_c}{P_c'}\right\}^2 \quad (9).$$

To these five relations we must, following Ref. 1 and 4, append two other empirical relations connecting certain pertinent variables to other important variables which can be scaled at will. The functions are

$$B_o = \frac{K_o}{\mu_o} \text{ and } P_c = \gamma j(S) \sqrt{\frac{\phi}{K}}.$$

From these we find:

$$\frac{B_o}{B_o'} \cdot \frac{\mu_o}{\mu_o'} = \frac{K_o}{K_o'} \quad (10), \qquad \frac{P_c}{P_c'} \cdot \sqrt{\frac{K_o}{K_o'}} = \frac{\gamma}{\gamma'}, \quad (11)$$

where μ_o = viscosity of oil phase, γ = interfacial tension between liquids. and K = specific permeability of sand.

The new variables entering Eqs. 10, 11, being arbitrary, permit a suitable model to be made. For example, the nature of available sand and fluids will give definite values to γ/γ', K_o/K_o', and μ_o/μ_o'. Clearly this will fix B_o/B_o' and P_c/P_c' so that only one other ratio may be chosen arbitrarily in Eqs. 5-9.

In Ref. 1,

$$\frac{\gamma}{\gamma'} = 2, \; \frac{L}{L'} = 16, \; \frac{\Delta\rho}{\Delta\rho'} = 1, \; \frac{g}{g'} = 1.$$

When we use these numbers in Eqs. 5-11 we obtain exactly the same numbers for the various ratios as were obtained in Ref. 1. Thus, the scale ratios used in Ref. 1 have been checked by use of the principle of dynamical similarity.

Current experimentation (Ref. 1) involves only six of the above pertinent variables, and $\Delta\rho$ and g are omitted. For this case we have:

$$[\pi] = B_o^t\, P_c^u\, Q_t^v\, q_t^w\, L^x\, \Theta^y$$
$$= (M^{-1} L^3 T)^t (ML^{-1} T^{-2})^u (L^3 T^{-1})^v (LT^{-1})^w L^x T^y.$$

Evidently we will obtain from these the set of equations (2) with $s = O$ and $z = O$. Solutions of these new equations are:

$$t = u = -2w + x + 3y \qquad (12)$$
$$v = w - x - 2y$$

By selecting (w, x, y) to be (1, 0, 0), (1, —1, 1), (1, 1, 0) we may find that (t, u, v) are respectively (—2, —2, 1), (—1, —1, 0), (0, 0, 0). These lead to the first three π's previously given and in the same order. Thus, again we may consider Eqs. 5-7 and Eqs. 10, 11.

Clearly the scale ratios of Leverett *et al.* (4) will satisfy this reduced set of equations, excluding use of ratios for g and $\Delta\rho$. Furthermore, the scale ratios chosen (1) are:

$$\frac{B_o}{B_o'} = \frac{1}{100}, \quad \frac{\gamma}{\gamma'} = \frac{16}{10}, \quad \frac{L}{L'} = 16, \quad \frac{q_t}{q_t'} = \frac{1}{100}.$$

The reduced set of equations then yields the same values of the other ratios as were found (1). Thus, the scale ratios used (1) have been verified.

The additional information available from the π-Theorem is that, if a functional relation does exist, it has the general form:

$$Q_t = \frac{B_o^2\, P_c^2}{q_t} F\left\{\frac{q_t\, \Theta}{L}, \frac{q_t\, L}{B_o\, P_c}\right\}. \qquad (13)$$

There seem to be reasonable grounds to consider q_t as being defined by L and Θ, so that π_2 does not enter the problem independently. If so, the relation above could be simplified to:

$$Q_t = \frac{B_o^2\, P_c^2}{q_t} F\left\{\frac{q_t\, L}{B_o\, P_c}\right\}. \qquad (14)$$

If sufficient experimental data is available it may be possible to determine whether this relation expresses some sort of "physical law". In any case Eq. 13 or Eq. 14 would assist in designing experiments, in which, for example, $\frac{q_t\, L}{B_o\, P_c}$ is varied systematically and the values of $\frac{Q_t\, q_t}{B_o^2\, P_c^2}$ might then define a set of values for the function in Eq. 14.

If dimensional analysis gives results not verified by experiment it is not the fault of the method but only of the faulty experimental information. This type of fault may be found with any theoretical investigation, however. A result may be good for one range of the variables but will fail for other ranges. This merely means that some new phenomenon is occurring which was not considered in the analysis. Dimensional analysis leads to the same type of results as a complete theoretical investigation but does not substitute for the latter. The latter method contains the mechanism of the phenomena, while dimensional analysis depends on empirical techniques.

LITERATURE CITED

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