Asymptotic Pursuit

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In the motion of a planet around the sun the pull of gravity on the planet is *toward* the sun, and since F=ma, the acceleration is likewise centripetal. But the velocity vector does not point toward the sun, since the planet has an angular momentum which causes a uniform areal sweep of the radius vector. In the military problem of guided missiles, the guidance mechanism may control the missile so that its velocity is always directed toward the target. It is this latter kinematical condition which is meant by mathematical *pursuit*. If P pursues Q the velocity of P is directed toward Q. To Pierre Bouguer in 1732 the point P was a pirate ship in pursuit of a merchantman Q which traveled a straight course. Bouguer computed the path of the privateer.

In 1906 Dunoyer supposed a fixed rudder might cause Q to move helplessly in a circle, but though his analysis occupies 29 pages in the Nouvelles Annales de Mathematique, he did not succeed in writing down the path of pursuit. This is not surprising since the differential equation in rectangular coordinates is

$$\frac{(1 + y'^2)^{3/2}}{my''} = 1 - \frac{yy' + x}{\sqrt{a^2(1+y'^2) - (y - xy')^2}}$$

In the Ladies' Diary for 1748 John Ash supposes that Q is a fly crawling around a semicircular pane of glass chased by a spider. In 1894 appeared an outdoor version with a jockey pursuing a horse that runs around a circular track. A more recent version of the problem appears in the American Mathematical Monthly for 1920. According to A. S. Hathaway of Houston, Texas, a dog makes straight for a duck which is swimming along the edge of a circular pond. If the dog swims m times as fast as the duck, the problem is to find the curve of pursuit which the dog follows. The equation of the dog's path is never written down, but Hathaway's five page study is the best published. Among the interesting results which he proves is that "capture can take place only from directly behind or directly in front." Let the duck's course be counterclockwise keeping the pond on the left. Then Hathaway says that if the dog is initially on the duck's right *capture can occur only from behind.* If this unchallenged statement were true, a dog which approached the pond from the far east would eventually find itself on the duck's right during the northward phase of the duck's course, and according to Hathaway capture could occur only from behind and not at all unless the dog's speed m is greater than the duck's speed. But this is not so. For a certain critical path of approach the duck will swim directly into the dog's mouth!

We show the existence of frontal capture by starting with the duck in the dog's mouth and running the cinema backwards: The dog backs directly away from the duck in a *curve of flight*. The distance of the dog from the pond will increase without bound, and we can show that it will approach a certain limiting direction. Let us choose polar coordinates (r, θ) with origin at the center of the pond and with polar axis extending in this asymptotic direction, which will depend on the relative speeds. Then we show first that r becomes infinite, and that θ becomes infinitesimal. Further, the flight curve never recedes from the polar axis by as much as the pond radius. The flight curve is a locally damped oscillation, but we do not know whether the polar axis is actually an asymptote.

To show that r becomes infinite consider the radial component of the dog's flight. The dog remains outside the pond, and while the duck is on the far side of the pond r is at least as great as the pond radius. The dog's speed is m and since the angle subtended by the pond is acute the radial speed is at least m cos45° half the time, with an average greater than .35 m. This means that r will eventually become larger than any bound.

To show that θ becomes infinitesimally small consider the direction of flight. From any point P on the flight curve draw the two lines tangent to the pond circle. The entire flight curve must remain in the angle containing the pond and its vertical angle. As r increases indefinitely these equal vertical angles become vanishingly small. In the language of limits, for arbitrarily small ϵ there exists a sufficiently large value of r beyond which the pond subtends an angle less than 2ϵ and so that the flight direction changes by less than ϵ. Therefore θ approaches a limit, which we have chosen as the polar axis.

That the dog never recedes from the polar axis by as much as the pond radius can be shown by indirect proof. For a contrary point, tan θ would be at least $1/r$ and would remain as large for future times. This contradicts our previously established result that the limit of θ is zero. Hence r sin θ is numerically less than the pond radius. If the speed m varies, this is the best result obtainable. But when m is a constant it seems likely that the limit of r sin θ may vanish. We have not found a proof of this conjecture.