A Graphical Solution of the System of Four
Linear Equations in Four Unknowns

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Consider the following set of four linear equations in four unknowns:

(1) $ax+by+cz+du=e$

(2) $a'x+b'y+c'x+d'u=e'$

(3) $a''x+b''y+c''z+d''u=e'''$

(4) $a'''x+b'''y+c'''z+d'''u=e'''$

Any two fo these equations determine a plane in 4-space. Consider the two planes determined by the first and the second pairs, respectively. If these two planes do not lie in the same 3-space, they have one and only one point in common, and the coordinates of this point constitute the solution of the system. If they lie in the same 3-space, then either they intersect or they are parallel, and the system has infinitely many solutions or none.

Therefore the problem of solving this system of four equations graphically reduces to the following problem: Given two planes in 4-space, construct their point or line of intersection, if such a point or line exists.

Before indicating this construction, certain preliminary notions have to be introduced:

A Mapping of the Euclidean 4-space, E_4, on the Plane E_2.

Consider a point $P(a,b,c,d)$ in E_4. Also, consider four parallel lines x,y,z,u in E_2 at unit distance from each other and another line in E_2 perpen-

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dicular to these lines. This line intersects the four parallel lines in four points O_1, O_2, O_3 and O_4. On the line x let O_1X be equal to a, on y let O_2Y be equal to b, etc. For every point $P(a,b,c,d)$ in E_4 we thus obtain four points X,Y,Z,U on x,y,z,u, respectively. These four points X,Y,Z,U are taken as the image of $P(a,b,c,d)$ in E_2.

Two other points $P(a',b',c',d')$ and $P(a'',b'',c'',d'')$ will go into two sets X',Y',Z',U' and X'',Y'',Z'',U''. If these three points are collinear, it can be proved that the lines XY, $X'Y'$ and $X''Y''$ are concurrent. Likewise, the lines YZ, $Y'Z'$, $Y''Z''$ and ZU, $Z'U'$, $Z''U''$ are concurrent. These points of concurrency, which we call xy, yz and zu are taken as the image of the line determined by the 3 points. It can be proved that every other point of this line will have the same property and every point having this property belongs to the line.

The images of a point, line, point on a line, and two parallel lines under this mapping, are given in Figure 1.
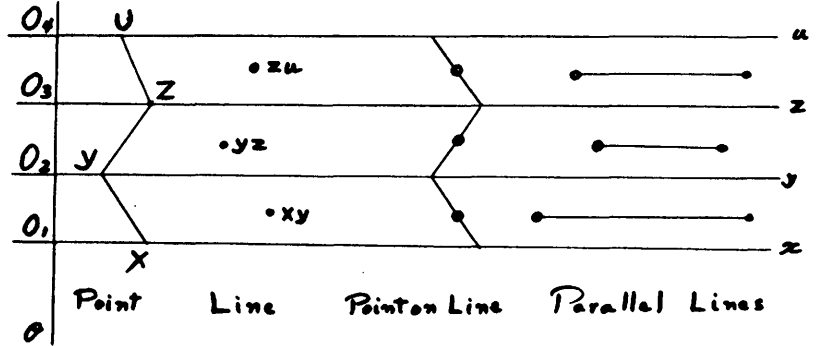

FIGURE 1.

A plane is determined by two intersecting lines and its image is given in Figure 2.
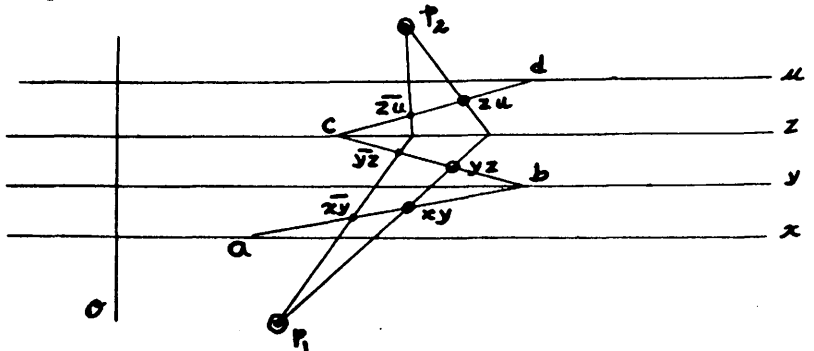

FIGURE 2.

The set p_1, p_2 (in Fig. 2), is also the image of a line of this plane since it has a a point in common with each of the two lines. This line (p_1, p_2) is called the axis of the plane. Every plane has one such axis.

Since a plane is also determined by one of its lines and one of its points not lying on the line, and since the axis of the plane is a line of the plane, a plane is determined by its axis and one of its points.

We now consider two such planes and find their intersections.

Consider two planes A and A' determined by their axes p_1 p_2 and p'_1, p'_2 and the points (a,b,c,d) and $(a'\ b'\ c'\ d')$, respectively. (Fig. 3).

In the plane E_1, draw two lines passing through p_1 and p'_1 and inter-secting ab and bc at the points xy and yz and $a'\ b'$ and $b'\ c'$ at the points $x'\ y'$ and $y'\ z'$ respectively, and with the additional property that the lines connecting xy to $x'\ y'$ and yz to $y'\ z'$ are parallel to y.
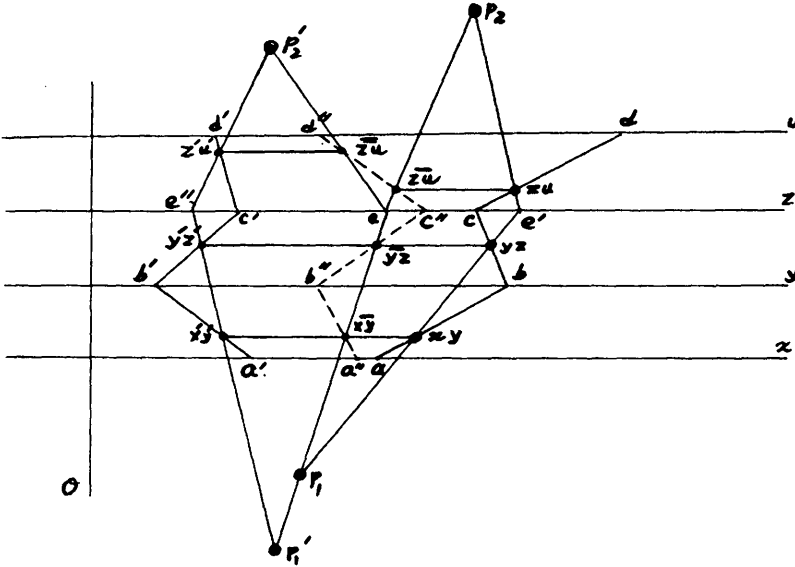
FIGURE 3.

This construction is always possible and unique.

Connect the point p_1 to p'_1 to determine the points $\overline{xy}$, $\overline{yz}$ and e. Connect the points e and e' to p_2 and e and e'' to p'_2 to determine the points zu, $z'u'$, $\overline{zu}$ and $\overline{\overline{zu}}$.

It will be noticed that the 3 points xy, yz, and zu constitute the image of a line in the plane A, and hence $\overline{xy}$ $\overline{yz}$ and $\overline{zu}$ also constitute the image of a line belonging to the plane A. ($\overline{xy}$, $\overline{yz}$, $\overline{zu}$ is parallel to one of the lines of the plane and has a point in common with the plane).

Similarly $\overline{xy}$, $\overline{yz}$, and $\overline{\overline{zu}}$ is the image of a line in plane A'. Now con-sider the point (a'', b'', c'', d''). It belongs to both these lines and hence to both planes, therefore, it is the point of intersection of the two planes.

In case the line connecting $\overline{zu}$ to $\overline{\overline{zu}}$ is parallel to the Z axis, the point (a'', b'', c'', d'') is a point at infinity.

In case the points e, p_2 and p'_2 are collinear and $\overline{zu}$ coincides with $\overline{\overline{zu}}$, the two planes have a line in common, namely the line whose image is $\overline{xy}$, $\overline{yz}$, and $\overline{zu}$. This happens in case the two planes lie in the same 3-space.

The Second Step in Solving the Problem

Given two linear equations 1) and 2) in four unknowns, find the image of the plane they determine.

To do this we find 3 points of the plane and map them on E_2. We then find the axis of the plane and consequently its graph.

Since there are 2 equations and four unknowns, it is possible to select two of the coordinates of the point arbitrarily and find the other two. In particular, these two coordinates are selected in such a manner that the computations will be simplified. For example, consider $x=o$ and $y=o$. Then

$$cz+du=e$$
$$c'z+d'u=e',$$

Hence

$$z=\frac{\begin{vmatrix} e & d \\ e' & d' \end{vmatrix}}{\begin{vmatrix} c & d \\ c' & d' \end{vmatrix}} \ , \quad u=\frac{\begin{vmatrix} e & c \\ e' & c' \end{vmatrix}}{\begin{vmatrix} d & c \\ d' & c' \end{vmatrix}} \ , \quad x=o, \quad y=o$$

So one point of the plane is

$$\left(o, \ o, \ \frac{\begin{vmatrix} e & d \\ e' & d' \end{vmatrix}}{\begin{vmatrix} c & d \\ c' & d' \end{vmatrix}} \ , \ \frac{\begin{vmatrix} e & c \\ e' & c' \end{vmatrix}}{\begin{vmatrix} d & c \\ d' & c' \end{vmatrix}}\right)$$

Another convenient point is the following:

$$ax+by+cz+du=e$$
$$a'x+b'y+c'z+d'u=e'$$
$$x=y$$
$$y=z$$

The solution of this system is:

$$x=y=z=\frac{\begin{vmatrix} e & d \\ e' & d' \end{vmatrix}}{\begin{vmatrix} a+b+c & d \\ a'+b'+c' & d' \end{vmatrix}} \text{ and } u=\frac{\begin{vmatrix} a+b+c & e \\ a'+b'+c' & e' \end{vmatrix}}{\begin{vmatrix} a+b+c & d \\ a'+b'+c' & d' \end{vmatrix}}$$

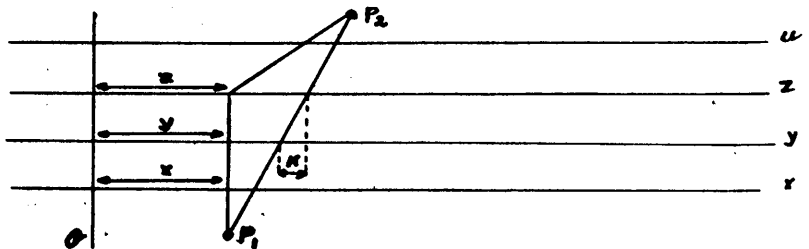The graph of this point is shown in Figure 4.

FIGURE 4.

FIGURE 4.

Finally consider the point determined by the following equations:

$$ax + by + cz + du = e$$
$$a'x + b'y + c'z + d'u = e'$$
$$y = x + k$$
$$z = y + k = x + 2k$$
$$u = z + k = x + 3k$$

These five equations give us:

$$x = \frac{\begin{vmatrix} e & b+2c+3d \\ e' & b'+2c'+3d' \end{vmatrix}}{\begin{vmatrix} a+b+c+d & b+2c+3d \\ a'+b'+c'+d' & b'+2c'+3d' \end{vmatrix}}, \quad k = \frac{\begin{vmatrix} e & a+b+c+d \\ e' & a'+b'+c'+d' \end{vmatrix}}{\begin{vmatrix} b+2c+3d & a+b+c+d \\ b'+2c'+3d' & a'+b'+c'+d' \end{vmatrix}}$$

$$y = x + k, \quad z = x + 2k, \quad u = x + 3k$$

and the image of this point is given in Figure 4.

It will be noticed from the figure that these two last points determine the axis of the plane. This axis and the first point determine the plane uniquely. And the rest of the solution follows.

To summarize the above results: Consider four equations. Take two of them at a time and plot them by finding 3 points of each plane. Construct the point of intersection of these two planes. The coordinates of this point, which can be measured, constitute the solution of the system. If the two planes are parallel or intersect the problem has none or infinitely many solutions.