THE HISTORICAL DEVELOPMENT OF TOTAL DIFFERENTIAL AS THE PRINCIPAL PART OF THE INCREMENT OF A FUNCTION OF SEVERAL VARIABLES*

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G. W. Leibniz. It is recognized that already in the seventeenth century the differential calculus had developed to such an extent that, according to Moritz Cantor, its only principal deficiency was a unified language and symbolism. However, its crystallization into a connected and effective system was, in the first place, accomplished by Leibniz in quite an independent and original manner which became a standard form for years to come.

There is no doubt that Newton possessed an analogous solution, perhaps somewhat earlier than Leibniz obtained his own; yet the fundamental term "Fluxion" advanced by Newton was, for the first time, made public in 1687—in his "Principia . . .", whereas its notation became generally known through a publication by Wallis only in 1693 and the method itself in Newton's "Quadratures of Curves" in 1704.

Earlier than these dates Leibniz had published his articles: "A New Method for Maxima and Minima . . ." in May and October, 1684 and "On the Abstruse Geometry" in June 1686, in "Acta Eruditorum". Thus the year 1684 became a landmark in the history of mathematics.

In the articles published that year Leibniz established both appropriate mathematical vocabulary and effective symbolism accompanied by immediate applications in the form of very useful and still usable operational rules called by him algorithms of Differential Calculus, further development of the theory of maxima and minima, and extension of differential equations to transcendental lines, etc. Also he generalized quadratures into the inverse methods of tangents, that is, into Integral Calculus, asserting that "sums and differences or $\int$ and d are reciprocals", in his article "On Abstruse Geometry . . ." and even much earlier in his manuscript of July, 1676 under the title "Inverse Methods of Tangents . . ."

All this is now recognized as the original and most fruitful major contribution to an unprecedented development of analysis during the eighteenth and nineteenth centuries. Leibniz, according to M. Cantor, had visions of this great future; he had, from the very beginning, recognized the significance of mathematical form while his rivals did not see or did not want to see it.

Likewise, W. W. Rouse Ball tells us that the fact that all results of modern mathematics are expressed in language invented by Leibniz has proved the best monument of his work. This coincidence of the conclusions of the two historians of mathematics whose attitudes toward Leibniz otherwise are somewhat different contributes to the impression of reliability and objectivity of their judgment.

Yet there is more than formalism (language and symbolism) in the differential calculus of Leibniz, that has irrevocably been assimilated by mathematics: this is his basic concept of the differential, in particular, and infinitesimals, in general, on which his system has been built.

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On close examination it has been found that:

1. What Leibniz calls "difference" in analysis is actually "differential" in the sense of *the principal part of an increment of a function;* for instance, see his letter to Wallis on March 30, 1699; also his articles of the years 1684 and 1686.

2. He is far from considering quantities infinitely small as tending to zero and becoming zero in the limit; he dissociates them from the doctrine of limit in this sense; see same letter and same articles.

3. He says in his *Reply to Nieuwentijt*: "By infinitely great and by infinitely small, we understand something indefinitely great or something indefinitely small, so that *each conducts itself as a sort of class, and not merely as the last thing of a class.*" (Italicized by the writer).

We can clearly see here the elements of neoclassical doctrine of infinitesimals such as *variability* ("*indefinitely* small") and *set* ("sort of class"). Likewise it is evident from his "Reply . . . . ." that Leibniz was awake to the importance of the concept of *continuum* for foundations of analysis. This vision did not deceive him.

4. "When we speak of infinitely great . . . or of infinitely small quantities . . . we mean quantities that are indefinitely great or indefinitely small, i. e. as great as you please or as small as you please *so that the error that any one may assign may be less than a certain assigned quantity.*" (Italicized by the writer). Same source: "Reply ...."

This manner of approach from a standpoint of assigned approximations will, more than two centuries later, be resumed and elaborated upon by Whitehead in "An Introduction to Mathematics." Referring to Weierstrass he interprets a derivative as a standard or goal of such approximations.

5. The use of $\frac{dy}{dx}$ for designation of a derivative was, with Leibniz, not only an improvement over Newton's fluxional notation ("dotism"), but its significance proved also to be operational, for it *formally* replaced the limit process by the elementary operation of division; it had tremendous repercussion on *Calculus as such*, bringing about an important simplification of its technique, to mention only, as an example, the setting up of differential equations. It was a real 'arithmetization" of calculus from the operational point of view. Now and then Leibniz interprets $\frac{dy}{dx}$ in terms of a limit, which is the evidence of his mathematical broadmindedness.

6. Leibniz admits that the doctrine of . . . "infinite extensions successively greater and greater or infinitely small ones successively less and less . . ." may be ". . . open to question . . ."; yet he continues: " . . . *it will be sufficient simply to make use of them as a tool that has advantages for the purpose of the calculation. . . For they contain a handy means for reckoning as can manifestly be verified in every case in a rigorous manner by the method already stated* (Italicized by the writer). See "Reply.........."

Time has brought full justification of these assertions and hopes entertained by Leibniz. Therefore his doctrine can not be all wrong; certainly there is in it considerably more truth than fallacy in spite of some obscurity and blur of its characteristics.

The Eighteenth Century. The first half of the eighteenth century showed deterioration of the foundations of calculus in the hands of followers both of Newton and of Leibniz, in puzzling contrast to an irresistible impetus of its growth and development of application. A balance was restored by

Lagrange when the eighteenth century had ended; the foundations of calculus were braced by him in a kind of synthesis of the two major doctrines: those of Leibniz and of Newton.

In the preface to the *second* edition of his Analytical Mechanics issued in 1811 Lagrange says: "When we have grasped the spirit of infinitesimal method and have verified the exactness of its results either by geometrical method of prime and ultimate ratios, or by the analytical method of derived functions, we may employ infinitely small quantities as a sure and valuable means of shortening and simplifying our proofs." (Quoted in W. W. Rouse Ball's "A Short Account of The History of Mathematics").

It has been seen that Leibniz was at times inclined to a similar synthesis.

The Nineteenth Century. A. L. Cauchy added much rigor to the treatment of both derivatives and differentials. Following in the footsteps of Leibniz and Lagrange Cauchy showed how total differential could be determined and found *independently of derivatives* as the *principal part* of the increment of a function, although he did not use this term. *K. Weierstrass introduced it.* For the modern doctrine of Analysis we are mainly indebted to him and Cantor as well as to Dedekind and Jordan.

Yet looking backwards we can clearly discern the great, although blurred outlines of modern calculus in the characteristics of Leibniz's doctrine as analyzed and put forth in the first article of this paper.

In the hands of Leibniz analysis became a particular form of calculus in full agreement with his *earlier far reaching* attempts to establish symbolic rules of operation for thought processes. Leibniz as logician and mathematician promoted one and the same form of structure: that of calculus both in logic and in analysis.

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