PARABOLAS INSCRIBED IN A TRIANGLE

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It can be shown analytically, according to R. A. Johnson, page 212 of his Modern Geometry, that the Simson lines of all points of a circle are tangent to a hypocycloid of three cusps circumscribed about the nine-point circle of the basic triangle. This result may be obtained by a very simple synthetic discussion. The Simson line of the point F of the circumcircle of the triangle passes through the point F′ midway between F and the orthocenter. As F and F′ describe homothetic arcs intercepted by equal central angles θ, the Simson line rotates uniformly through an angle $-\theta/2$. The identity of its envelope with a hypocycloid of three cusps is established as follows. Let O be the center of the fixed circle, Q the point of contact between the rolling circle and the fixed circle, and R the point of the rolling circle on the line OQ. Let S be the corresponding point of this hypocycloid. Then QS is normal to the curve and RS is tangent. Also call the initial point of contact X, and let angle XOQ be θ. We see readily that angle QRS is $3\theta/2$ and that RS makes an angle $-\theta/2$ with OX.

An interesting problem which is handled as a direct application of this theory is encountered upon considering all parabolas tangent to the sides of a given triangle. It is well known that the focus lies on the circumcircle of the triangle and that the tangent at the vertex is the Simson line of the focus, so we conclude: *The envelope of the tangents at the vertices of all parabolas inscribed in a given triangle is a hypocycloid of three cusps circumscribed about the ninepoint circle.* Since the axis of a parabola is perpendicular to the tangent at the vertex and passes through the focus, we have the theorem: *The envelope of the axes of the family is a hypocycloid of three cusps circumscribed about the circumcircle.* Attention should be called to the fact that the directrices of all the parabolas pass through the orthocenter.

It is of interest to see if there is a simple construction for the cusps of the hypocycloid. If H is the orthocenter of the given triangle ABC, let K, L, M be the points where RA, RB, RC cut the ninepoint circle, and let radii of this circle parallel to these altitudes meet the circle in P, Q, R. Then the cusps are on the radii which divide the arcs KP, LQ, MR in the ratio 2 to 1. To see this, we need to remember that the altitudes are Simson lines, and the Simson line rotates half as fast in the opposite sense to the corresponding radius of the ninepoint circle.

If the triangle is equilateral, the figure takes on a particularly pleasing appearance, and we may also state the following metric property: If the vertices A. B, C are on a circle of radius unity and have the position angles $\pi/3$, π, $-\pi/3$ respectively, and if the focus has the position angle θ, then the distance from the focus to the directrix is $\sin(\pi-3\theta)/2$. The same magnitude may be given in the form $2(FL^2 \cdot FM^2 \cdot FN^2)^{1/3}/(FA \cdot FB \cdot FC)^{1/3}$, where L, M, N are the feet of the perpendiculars from the focus F to the sides of the triangle. This relation is obtained from the three proportions like FX:FL::FN:FC, where X is the orthogonal projection of F on its Simson line. These arise from the similarity of pairs of triangles such as FMN and FCB.

The familiar triangle theorem of projective geometry asserts the concurrence of the lines joining the vertices of a triangle circumscribed about a conic to the contacts on the opposite sides. If the triangle is equilateral and the conic is a parabola, the common point is the focus, a result easily obtained from the theorem that the line from the point to the intersection of any two tangents bisects one of the angles formed by the focal radii of their contacts. This property furnishes a very simple construction for the focus of a parabola given by three tangents forming an equilateral triangle and the point of contact on one of them.

By parallel projection of this figure on a new plane, we can conclude that for the general triangle the locus of the point of concurrence in the triangle theorem is an ellipse passing through the vertices of the triangle.