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PASCAL'S LAW IN FLUIDS IN RADIAL FIELDS OF FORCE

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Pascal's law states that pressure in a fluid is transmitted undiminished in every direction. It is fundamental in the statics and dynamics of fluids and therefore is stated in every elementary text book. Nevertheless, even in more advanced treatises, it is seldom, if ever, applied to fluids in other than sensibly parallel fields of force. Thus Kelvin and Tait in their "Treatise on Natural Philosophy," article 752, state without proof that "The rate of increase of pressure per unit of length in the direction of the

resultant force is equal to the intensity of the force reckoned per unit of volume of the fluid." This clearly includes the general case of fluids in non-parallel fields, yet the same authors afterwards apply it only to the special (but important) case of parallel fields.

Some interesting results, however, may be obtained in an elementary manner by applying Pascal's law to fluids in radial fields of force, as will now be shown.

The problem of the pressure generated by a centrifugal water pump may be simplified by assuming that the angular velocity of the water is everywhere the same. Then by Pascal's law the pressure on the inner arc of an arbitrary infinitesimal of fluid mass $\rho r dr d\theta$ is transmitted undiminished to the outer arc, the total pressure at which will therefore be the pressure at the inner arc plus that due to the centrifugal reaction of the whirling element of mass. That is, the pressure at any point distant r from the center will be given by the integral

$$P = P_1 + \int_{r_1}^r (\omega^2 r) (\rho r dr d\theta) \left(\frac{1}{r d\theta} \right) \quad \text{Eq. (1a)}$$

$$= P_1 + \frac{1}{2} \rho \omega^2 (r^2 - r_1^2),$$

where P_1 is the pressure at r_1 , $\omega^2 r$ is the force per unit mass due to the centrifugal reaction and $\rho r dr d\theta$ is an element of mass. If both

$$P_1 \text{ and } r_1 \text{ are zero, } P = \frac{1}{2} \rho \omega^2 r^2. \quad \text{Eq. (1b)}$$

We could have obtained the same result by integrating directly the mathematical expression for the statement above by Kelvin and Tait.

$$\frac{dp}{dr} = \text{force per unit mass} = \omega^2 r.$$

Or the same result could be obtained starting from the hydrodynamical equation of continuity,

$$\frac{d\rho}{dt} = -\rho \left(\frac{\delta V_x}{\delta x} + \frac{\delta V_y}{\delta y} + \frac{\delta V_z}{\delta z} \right)$$

as shown in Page's "Introduction to Theoretical Physics."

It is instructive to notice that the expression which we have obtained for the pressure gives for the total force sustained by the outer (cylindrical) wall of the pump a quantity greater than the centrifugal force integrated over all the elements of mass in rotating fluid. Thus if we had naively attempted to compute the pressure on the outer wall by assuming that the total force supported by it was the summation of all the centrifugal forces acting on the fluid particles, we would have obtained

$$P = \frac{1}{2 \rho a} \left[(2 P_1 r_1 P_1 + \int_{r_1}^a (\omega^2 r) \rho r dr d\theta) \right]$$

$$= \frac{r_1 P_1}{a} + \frac{1}{3} \frac{\rho \omega^2}{a} (a^3 - r_1^3) \quad \text{Eq. (2a)}$$

which for P_1 and r_1 both zero and $a = r$ becomes $P = 1/3 \rho \omega^2 r^2$. Eq. (2b). The incorrectness of this method of procedure is immediately apparent from (2a), which for the case of $\omega = 0$ reduces to $P = r_1/a P_1$. This in turn

for vanishing r , would make the pressure P zero, regardless of the value of the initial pressure P_1 —which is plainly a contradiction to experience, a contradiction recognized, in fact, by a simple application of Pascal's law.

The expression of Eq. (2b), however, if multiplied by the area of the external supporting wall, will give the total integrated radial force acting on all the fluid elements, as may be seen by the way in which it was derived. But this is less than the actual total force acting on the supporting wall, which is given by multiplying the correct pressure $\frac{1}{2} \rho \omega^2 r^2$ by the area. The total force on the outer wall is therefore greater than the total force acting on the water, in contrast to the usual case in a parallel (gravitational) field, in which the resultant (vector) force acting on the walls of a vessel is equal to the total weight of the fluid contained in the vessel.

In this case of what we may call a diverging field of force, the total force acting on a given supporting surface is *greater* than the integrated force acting on all the volume elements. Likewise in a converging radial field, such as the gravitational field of the earth, the total force on a given supporting surface is *less* than the integrated force acting on all the volume elements.

The mass of the atmosphere is usually computed by assuming that it is equal to the product of the earth's surface by the pressure (measured in grams weight or pounds weight) per unit area due to the atmosphere. But this does not give the total mass. If the operation of Pascal's law in a radial field is applied to this problem, a value for the mass of the atmosphere will be obtained which is approximately 0.33 per cent greater than the value obtained by making the usual approximations (which amounts to assuming that the earth is flat).

The correction here involved is probably not large in comparison with other errors introduced by neglecting the effects of the earth's rotation and temperature gradients. It is interesting theoretically, however, inasmuch as it shows that the total weight of the atmosphere is not supported by the earth's surface, but that part of the atmosphere—about 0.33 per cent—is supported by itself, somewhat on the principle of the arch, i. e., in the same manner that we might imagine a solid ring encircling the earth at the equator to support itself by compressional forces, rather than by resting directly on the earth's surface.

