

## C. PHYSICAL SCIENCES

### I. ON THE CONCEPTS OF MASS AND FORCE

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The concepts of Mass and Force are of fundamental importance in Physical Science. They are, however, rather difficult concepts, and their definitions are given very unsatisfactorily in most textbooks—not only in elementary books but even in many advanced books in mechanics. The difficulty is due not only to the fact that the fundamental laws of dynamics do not lend themselves well to demonstration by simple experiments, but also to the fact that the two concepts have not been developed independently. There is, at least, historical and pedagogical truth in what A. N. Whitehead says in his book "The Principles of Natural Knowledge": "We obtain our knowledge of forces by having some theory about masses and our knowledge about masses by having some theory about forces."

Force is an older concept than mass. It has its origin in our sensations of push and pull, and was first developed as a scientific concept in statics or the theory of equilibrium of bodies. A body **A** resting in contact with another body **B** is said to exert a force on **B**, if the removal of **A** will result in a change of position of **B**. It is a matter of experience that the same effect on **B** can always be secured by application of strings, pulleys and weights or by the use of elastic springs, and this fact provides the methods of measuring force. The force of **A** upon **B** is either measured by the weight which must be applied to keep **B** in equilibrium after the removal of **A**, or by the stretch or shortening of a spring used for the same purpose. Force is a vector quantity, since the spring or the string carrying the weight must have a definite direction. This static idea of force is quite independent of any notion of mass.

In the Science of Dynamics, which has as its purpose the description of the motion of bodies, the concept of force is generalized, and the idea of mass introduced. The realization of the necessity for a concept of mass, different from weight, and the presentation to us of a dynamical concept of force, are, perhaps, Newton's greatest achievements. However, Newton did not define these concepts as well as he applied them. This is quite natural. It would indeed have been remarkable, if the marvelous development of Physics since Newton's time had called for no modification of his statement of the fundamental principles of

dynamics. The peculiar thing is that more than 250 years should pass before a constructive criticism was given of Newton's definitions of mass and his three laws of motion, and that even today the effect of this criticism is hardly noticeable in the textbooks. Newton defined mass as "quantity of matter" as obtained by multiplying density by volume. This of course is not a definition of mass but rather of "quantity of matter" or "density." A real definition of mass is only implicitly given in Newton's works. But a great number of texts still define mass as "quantity of matter." A book used last year in a course in analytical mechanics in the University simply states mass is "measured matter." Books of this type give, in general, no dynamical definition of force but carry the concept over from statics without any discussion of the difficulties involved, or they define force as the "cause of a change of motion." The latter definition is very common indeed, but in the first place it is not quantitative and in the second place it is metaphysical in presupposing the idea of "efficient causation" which has no place in Science.

Other books define mass as the weights found by means of a beam balance, or as the ratio of the weight measured by a spring balance to the acceleration of gravity. These definitions are based on the proportionality of mass and weight, a fact, which, however important and remarkable it may be, has nothing to do with the fundamental laws of dynamics. Such texts do not define mass as a dynamical concept, but merely give important methods for measuring it.

Before I give what I consider the only quite satisfactory definitions of mass and force, let me mention the best treatment of the subject that I have so far found in any elementary text. In Duff's "Textbook of Physics" a preliminary idea of force is first given. The ratio of the masses of two particles is then defined as the inverse ratio of the accelerations which equal forces will impart to the particles. Finally, force is defined as the product of mass and acceleration. The objection to this procedure is that it presupposes the possibility of defining equality of forces acting on different bodies. Now, the only methods of comparing forces, before the concept of mass is introduced, would depend on either gravitation or elastic properties, and that means that this way of defining the dynamical concepts of mass and force contains a non-dynamical element.

A purely dynamical definition of mass was first given by Ernst Mach in 1867. His paper was rejected by the *Annalen der Physik* but published a year later in a journal of less repute. Kirchhoff in the first volume of his famous "Vorlesungen über

Mathematische Physik" which appeared in 1874 gave a similar definition. Mach's criticism of the traditional treatment of the laws of motion was elaborated in his big work on the development of mechanics which came in 1883. Karl Pearson in his "Grammar of Science" which appeared in 1892 independently arrived at conclusions similar to Mach's. The new treatment of the fundamental concepts and laws of mechanics is adopted in such works as Boltzmann's "Lectures on the Principles of Mechanics," and Whittaker's "Analytical Dynamics," but other authors such as Hass in his "Introduction to Theoretical Physics" and Jeans in his "Theoretical Mechanics" follow more closely the older tradition.

Let us now consider a system consisting of any number, say,  $n$  material particles. It is a matter of experience that the acceleration of any particle may be resolved into  $n-1$  components which are directed towards each one of the  $n-1$  other particles. The acceleration  $a_{hk}$  of particle numbered  $h$  due to the presence of particle number  $k$  is found by experience to be independent of the positions or velocities of any of the particles as well as of the time, but to depend only on the distance  $r_{hk}$  between particle number  $h$  and particle number  $k$ . The acceleration  $a_{kh}$  which the  $k$ 'th particle receives due to the presence of the  $h$ 'th particle is always found to be in the opposite direction of  $a_{hk}$ , and the ratio between the two accelerations is found to have the same constant value at all times and for all distances of the two particles. Furthermore, it is found by experience that

$$(1) \quad \frac{a_{hk}}{a_{kh}} = \frac{a_{hl}}{a_{lh}}$$

where the index  $l$  refers to any other particle. All these experimental facts are expressed by the following equations.

$$(2) \quad m_h a_{hk} = -m_k a_{kh} = f_{hk}(r_{hk}); \quad h, k = 1, 2 \dots n$$

where  $m_1, m_2, \dots, m_n$  are constants, one of which may be chosen arbitrarily. The constant  $m_h$  is called the mass of the  $h$ 'th particle, and the product  $m_h a_{hk}$  is called the force exerted by particle number  $k$  upon particle number  $h$ .

This method gives a purely dynamical definition of mass and force. Equation (2) contains the law of Action and Reaction as well as the law of the Parallelogram of Forces. Together with the empirical result, that all the functions  $f_{hk}$  tend to vanish

as the  $ra$ 's approach infinity, it gives the Principle of Inertia.

I have not mentioned the question as to what frame of reference should be used in defining accelerations. For Newton this question did not exist, as he believed in an absolute space as well as in an absolute time. We have tacitly assumed that the coordinate system used belongs to the class of systems called inertial systems. If another type of coordinate system had been used the experimental result could not have been stated in so simple a form.

The definition of force as the product of mass and acceleration has been criticized by various writers. A. N. Whitehead, for example, writes in his "Principles of Natural Knowledge": "The difficulty to be faced with this definition is that the familiar equation of elementary dynamics, namely,  $ma = F$ , now becomes  $ma = ma$ . It is not easy to understand how an important science can issue from such premises." To this argument we simply remark that we have not claimed that anything should follow merely from the definition of force. The Science of Dynamics is based on the discovery that the product of mass and acceleration, or the force, is equal to a function of the distances between the particles, as expressed in equation (2). It is this fact which gives the product of mass and acceleration its great importance and justifies the introduction of a special name for it. Perhaps it would have been a little more satisfactory to define force not simply as the product of mass and acceleration but rather as the function of the distances between the particles to which this product is equal.

The dynamical concept of force is free of the anthropomorphic element which attaches to the static concept. The former is, of course, a generalization of the latter, since all statical phenomena as is easily shown, can be described by the laws of dynamics. While the old static concept of force thus has been replaced by the more comprehensive dynamical concept, it may be good pedagogy to make use of the static concept in elementary teaching to prepare the way for the more abstract dynamical concept. Mass is a purely dynamical concept, and should be defined as such.

I have considered the concepts of mass and force only from the point of view of classical mechanics. In the Theory of Relativity the concepts are redefined, and the Quantum Theory seems to call for a further modification. A discussion of these more difficult matters, however, is beyond the scope of the present paper.